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2. Used labels

We are specifying here only the labels used in scripts systematically and repeatedly. Other labels will be explained during their use.

Latin alphabet

\( D(\kappa) \) dissipation spectrum
\( e \) unit vector
\( E(\kappa) \) energy spectrum
\( f \) longitudinal correlation function (non-dimensional); frequency
\( g \) transverse correlation function (non-dimensional)
\( i \) imaginary unit
\( k \) kinetic energy
\( l \) length scale
\( L \) Integral length scale
\( p \) pressure
\( \text{Re} \) Reynolds number
\( s_{ij} \) strain-rate tensor
\( t \) time
\( u \) velocity vector
\( x \) position vector

Greek alphabet

\( \gamma \) intermittency coefficient
\( \delta \) Dirac function
\( \delta_{ij} \) Kronecker delta
\( \varepsilon \) dissipation rate
\( \varepsilon_{ijk} \) Levi-Civit permutation tensor
\( \eta \) Kolmogorov scale
\( \theta \) Heaviside function
\( \kappa, \kappa \) wave number, wave number vector
\( \lambda \) Taylor micro scale
\( \nu \) coefficient of kinetic viscosity
\( \rho \) density, correlation coefficient, auto-correlation function
\( \tau \) time scale
\( \zeta \) enstrophy
\( \omega \) vorticity vector

Indices

\( i, j \) tensor indices
\( k, l, m \) reading indices

Other symbols

\( a \) scalar “a”
3. Introduction

With the development of time-resolved methods for flow measurement, point, plane and volumetric, researchers and scientists have found that there is no such thing as “steady flow”, and that practically every flow is highly fluctuating.

The “statistic” approach to evaluation of measured fluctuations has very often resulted in characterizing such fluctuations by a single number, “Turbulence intensity [%]”. But these fluctuations, when measured by sufficiently fast time-resolved methods, like CTA or Time-Resolved PIV, carry a wealth of very important information about the flow and its interaction with the outside world.

To measure turbulence and not to understand and evaluate it properly means a net loss of time and resources. And this is why this text is being offered to you.

The text provides material for introduction to the study of turbulence in fluid flows. It contains an overview of basic information about “turbulence”. Although it contains a lot of equations and can be studied “seriously”, perhaps even with the help of literature specified in the References, it can be also read “less seriously”, just more or less skipping equations and mathematics, and it will still offer a reader a good overview of the subject, and enable the reader to understand what turbulence is about, what are its basic assumptions, what computer simulation folks are speaking about, why they often get less-than-perfect simulation results, and why experimental measurement of fluid flows will always be necessary to validate computer simulation assumptions.

Katsushika Hokusai, Large turbulent wave, 1829–32
3.1. Introduction to the study of turbulence

The flow of water in the river, clouds in the sky, burning flames, the starry universe – these are some examples of phenomena that we can label as turbulent. Turbulence has always been a fascinating phenomenon for people, even though (or indeed because) it is difficult to grasp, due to its variability and complexity. Ever since ancient times, thinkers have attempted to come to terms with the existence of turbulence, and this effort continues to this day. The process of recognizing the laws of turbulence has not been finished. It has been accompanied by unexpected consequences for various areas of science.

Figure 3.1 – “Snapshot” of water flow to a tank, Leonardo da Vinci, circa 1500

One of the first known findings about the structure of turbulence in modern times was the observation of fluid flows by Leonardo da Vinci – see Figure 3.1. Leonardo illustrated the flow of water as a moment when the turbulent flow field is comprised of various structures of various sizes. It has a very balanced effect with a complex and obviously consistent structure.

Another historical example of a regular structure in turbulent flow is the known red spot on Jupiter in Figure 3.2. It is basically an enormous storm – turbulent vortex (anti-cyclone) and has lasted at least 350 years (in 1655 it was first observed by French astronomer Cassini). It is more or less a stable storm, which, however, has been changing its position and structure in time relatively quickly. The existence of this regular structure in developed turbulent flow has not only captured the attention of astronomers, but of experts regarding fluid mechanics as well. The phenomenon has initialized a discussion regarding the structure of turbulent flows.

Figure 3.2 – Red spot on Jupiter
Everyday experiences enable us to recognize turbulence. The smoke that rises from a cigarette or fire shows the irregular behavior of the moving air that carries it. Wind is subject to sharp local changes in direction and speed, which can have dramatic results for sailors and pilots. During transport by passenger aircraft, the term “turbulence” is often associated with buckling seatbelts. The term is also used when describing free streams and streaks. When water flows in a river, its presence has an important effect for the settling of sediment on the bottom. Quick flow of fluid around an obstacle or around an aviation profile creates turbulence in the between layer and creates a turbulent jolt causing increased resistance strength, which causes the flow fluid to affect the obstacle (it is usually expressed with the help of the famous coefficient \(C_x\)). In order to achieve better aerodynamic properties of cars and aircraft, turbulence must be suppressed. The behavior of most oceanic and atmospheric flows cannot be exactly predicted, because they fall into the category of turbulent flows, and the same applies to flows of planetary scales. Small-scale turbulence in the Earth atmosphere represents a serious problem during astronomical observations conducted from the Earth surface, and it is a decisive factor to take into consideration when selecting an observatory. The atmospheres of planets such as Jupiter and Saturn, the solar atmosphere and the Earth outer atmosphere are turbulent. Galaxies typically have the shape of vortices (see the example in Figure 3.3) similar to those that occur in turbulent streams, such as flows in a mixing layer of two streams of different velocity. These are formed as a result of turbulent phenomena. We can name a lot of other similar examples from aerodynamics, hydraulics, nuclear and chemical engineering, oceanology, meteorology, astrophysics, cosmology or geophysics. On the opposite field of the spectrum there are quantum vortices occurring in a superfluid fluid, which have dimensions expressible in multiples of the average size of an atom. The realm of turbulence therefore includes our observed universe, and turbulence is a typical kind of behavior of that universe in all of its degrees.

![Figure 3.3 – Spiral galaxy in the Andromeda constellation](image)

### 3.2. Turbulence in the context of modern science

Modern science and modern physics are based on an approach that Isaac Newton discovered in the 17th century. The scientific method can be simplified for the application of the following three consecutive steps. The first step is analysis of the problem, which is conducted with the help of experiments, actual, physical or thought-based. The second step is synthesis, which involves a transition from a specific to a general action. This practically means that a mathematical model of the examined phenomenon is created. The third step is a prediction, which represents verification of a mathematical model and its application for specific situations, which are not identical to the situations considered in the first step (for these situations, the model must of course also apply).
The issue of turbulence somehow did not fall into the concept of science as defined by Newton, and the behavior of objects during a state of turbulence was not predicted reliably. In the past, turbulent behavior was often associated with magic, and to this day this issue is engulfed by a veil of secrecy. This problem has been dealt with systematically by a large number of scientific experts (see the addendum for a detailed overview), but practically all identified individuals involved in the study of physics have examined this problem at least to a certain extent. It is said about the Father of quantum theory, Max Planck, that on his death bed he declared that as soon as he appeared before God, he would have two questions: “Why quantum and why turbulence? I am confident that I will receive a satisfactory answer to the first question.” The problem of turbulence is often referred to as the last unresolved problem of classical physics.

A typical approach to creating mathematical models of natural phenomena is the reduction of the number of degrees of freedom and linearization. This approach leads in many cases to a relatively simple mathematical model, which can be analyzed successfully and applied to specific situations, often using an analytical method. The physical problem can be deeply examined with the help of reductionism, when the object of examination, in this case a material, is analyzed to the smallest possible details. If this is successful, the problem is declared solved, at least from a theoretical point of view. The holy grail of physics recognizing this reductionist approach is the “theory of everything”, which is defined as discovering truly the basic elements of material and examining its properties. The gradual precursors for this theory were the theories of atoms, elementary particles, quark theory and more recently the theory of strings and super strings.

It is becoming apparent, however, that reductionism does not resolve certain phenomena, among which we can include turbulence in fluids. The problem is that the very basis of turbulence is connected with systems with a huge number of degrees of freedom, which do not include non-linear links. When examining this phenomenon, we must approach fluid as a system of simple elements, whose properties are important for the behavior of the whole, but in relation to which a certain role is played by legitimacy, which is not possible to derive from the behavior of a single element. However, there is an even deeper cause of the failure of the reductionist approach, which is related to the separability principle, according to which the system can be examined by examining its parts separately. In the 1930s, this principle was the subject of passionate philosophical discussions. A supporter of the separability principle was A. Einstein, and N. Bohr was an opponent. Bohr was eventually proved right. He claimed that this principle generally does not apply and that the behavior of the whole cannot be expressed as the direct sum of the behavior of its parts.

There is nothing left to do but to give into the reductionist approach and try to create a new “holistic” (from the Greek holos – whole) approach involving examination of the system as a whole. Unfortunately, mathematical tools based on a differential and integral calculus, which we currently have available, were created for reductionist analyses, and their use in a holistic method of examination is very limited. They are not suitable for a description of phenomena such as complexity, fractal geometry, deterministic chaos and self-organization. New fields that deal with these phenomena (the theory of dynamical systems, the theory of catastrophes, artificial intelligence and last but not least turbulence) need to look for other, better suited ways to describe the situation.

3.3. Flowing fluid as a dynamical system

From the theory of dynamical systems, it is known that many actually connected systems can be modeled with a system of partial and differential equations, which then must be resolved numerically. Such dynamical systems theoretically have a never-ending number of degrees of freedom, and for their resolution initial conditions are required that are characterized by a never-ending number of conditions located in the particular space. The dynamic behavior of spatially connected systems can be variable both in space and in time, as a result of which both regular and chaotic structures can occur.

1 This conclusion has certain fatal consequences, such as the invalidity of determinism in its classic form.
The flow of fluids can be qualitatively characterized as laminar or turbulent. Laminar flow is typically either a very slow motion or involves a level of viscosity. Fluid particles move evenly and slide across each other in layers (lamina is Latin for layer, plate), and are therefore laminar. However, turbulent flows (turbulentus is Latin for uneven) are characterized by quick motion or a low effect of viscosity, when even minor perturbations in stream grow uncontrollably and cause unpredictable local behavior of fluid and intensive eddy mixing in the whole area (a more exact definition of turbulence will be presented later).

It is a certain paradox that the flow of entirely inviscid fluid is always laminar, and turbulence cannot occur in inviscid fluid. This is because fluid particles affect each other only through pressure forces, and tangential forces are zero. Swirling, which is characteristic for turbulent flows, cannot occur in such fluid. This fact is described in Thomson theorem about vortices, which for an ideal inviscid fluid is the equivalent to saying: “if the motion was not swirling at a certain moment, it will continue not to be swirling.” However, each real fluid is characterized by non-zero viscosity, and inviscid fluid is only an idealized limited case.

During laminar flows, the image of flow does not change in time, if the boundary conditions are also unchanging and if we detach them from microscopic phenomena such as Brownian motion. Therefore, we can claim that in this case there is no active dynamic degree of freedom of the fluid system. In the Lagrange sense\(^2\) each fluid particle, however large, behaves dynamically – it changes its position in time and space. However, if we look at the system as a whole from Euler’s perspective\(^3\), then we can regard the system as static.

A change of the flow conditions, usually an increase in velocity, can make the system unstable. Boundary conditions determine the new temporarily stable condition of the system, to which its development is directed – attractor. Immediately after the loss of stability, the behavior of the system is relatively simple, and there is only a very small number of active degrees of freedom, shifts from an unstable balanced condition are minor, and the behavior of the system can be described with sufficient accuracy using a linearized model. An increase in shifts leads to application of non-linearities, and the structure is increasingly more complicated and less evenly arranged and can even lead deterministic chaos and turbulence. In this condition, the number of active degrees of freedom is based on the character of perturbations that occur in the stream. This is an extended system, when the mutual correlation of the changes in the condition in two points with remoteness quickly drops to zero.

The fluid system in the state of turbulence constantly changes its structure in time. Nevertheless, for description of turbulent flow, models exist, which are used in relation to a statistical description with the help of statistic moments. These are models based on Reynolds’ equations and are identified in literature as RANS\(^4\) models.

Generally, the study of dynamical systems is carried out in a phase space, the dimensions of which are based on the number of degrees of freedom and the number of independent variables, which perfectly describe the condition of the system. Each point in the phase space represents a certain dynamic condition of the system.

A basic characteristic of the dynamical system is the attractor. This is a phase portrait of the condition to which the system is “attracted” (hence the name) during initial conditions lying in a certain area of phase space. The development of the system in time leads to this condition, and after a sufficiently lengthy amount of time it is approached without limitation. From the classical theory of dynamical systems, we are accustomed to two types of attractors. These can include the only point in the phase area, and the system also leads to a certain static and in time unchanging balanced condition. Another possibility is the limit cycle. This is a closed curve in the phase space, which characterizes the periodic change of the condition of the system. However, there is also a third possibility, which is the creation of a “strange attractor”, which is typical for the system condition known as “deterministic chaos”. This condition is typical for the turbulent behavior of fluid and has key importance for the study of fluid systems. We will explain the term “deterministic chaos” in greater detail in later chapters.

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2 Lagrange’s approach monitors the behavior of model fluid particles (see Chapter 4.2)
3 Euler’s approach is represented by a description of the system in a firmly fixed point in space (see Chapter 4.2)
4 Reynolds Averaged Navier-Stokes (equations) – see Chapter 9
The “bifurcation point” characterizes the condition of the system at the stability limit of a certain structure. Further development of the system can continue according to multiple scenarios. There can be two or more, or there can be an unlimited amount. The selection of the correct specific scenario depends on Brownian motion events, and from the point of view of anthropomorphic scales there is no remaining option other than to consider this process to be random, when the probability of the realization of individual scenarios is not zero, but is less than 1. At the end of each scenario, there is a new stable system condition or another bifurcation point.

### 3.3.1. Fractal structure

The dimensions of a geometric object are usually defined as the number of independent directions of motion of the point in its framework. In this case, it involves topological dimension $d_T$ always a natural (positive, whole) number. It can also be equal to or less than the dimension of the space $d$, in which the object is located. Although a smooth line and random trajectory have the same topological dimension $d_T=1$, they have very differing properties, because the random trajectory can entirely or partially fill a certain two-dimensional or three-dimensional sub-space. Therefore, fractal dimensions of the geometric object $d_f$ are introduced.

![Figure 3.4 – Example of a Mandelbrot aggregate (fractal)](image)

In 1975, Benoit Mandelbrot defined “fractal” (from the Latin *fractus* – broken) as an aggregate whose fractal dimension is greater than a topological dimension and is expressed by a non-whole number. It can also be defined somewhat more simply and less generally as a geometric object with the following properties: it is self-similar – this means that if we see a particular section in any scale or resolution, we will observe a constantly repeating and certain characteristic shape; and at first glimpse it has a very complex structure, but it can be generated by repeated use of simple rules (recurrence). Fractals are the most complex geometric object examined in current mathematics, but they often have a surprisingly simple mathematical structure. An example of an “artificial” fractal is shown in Figure 3.4.

The fractal structure is typical for all natural objects for which non-linear behavior is applied in combination with a large number of degrees of freedom. Natural fractals contain an element of randomness related to the occurrence of repeating structures at individual levels, which dramatically increases the number of degrees of freedom of such an object. A typical example is flow fluid. An example of a natural fractal structure is shown in Figure 3.5.

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5 The term “anthropomorphic” represents the point of view through the eyes of a person who through senses is capable of perceiving only phenomena whose spatial and time scale does not differ much from the scales of the human body and the motion that goes on in it. Therefore, if a person’s size is regularly a meter, then the anthropomorphic scales can be characterized based on a difference of 5 orders of magnitude, meaning $10^{-5}$ m to $10^{5}$ m, in the time area possibly from $10^{-3}$ s to $10^{9}$ s.
The Platonic world of simple geometric sections (type of spheres, boxes, polyhedrons,…) described in classical teaching materials emerged thanks to linear mathematical models, and even relatively complicated systems can report only a very low number of active degrees of freedom. However, we live in a non-linear world of fractal geometry.

For examination and analysis of fractal objects, methods developed for simple objects generally cannot be used. Characteristics such as the length of a line or the contents of a surface do not have practical significance for fractals, but the fractal dimension is key information. It is also apparent that for fractals that report a fractal structure for random small details, methods using a differential number cannot be applied, because derivative is not defined in relation to them.

3.3.2. Deterministic chaos

The term “chaos” generally refers to behavior that is an expression of absolute and pure randomness and in which there is no room for effects of orderly rules. Such behavior can be called absolutely incoherent, since there are no legitimate links between neighboring conditions (both in space and in time). The chaotic behavior of actual systems in nature is characterized by the term “deterministic chaos”. This is a process of self-organization of complex systems, when systems of coherent structures are created that behave according to lateral laws. From the point of view of one specific structure, the development of the system has elements of randomness, but the system as a whole is developing entirely according to rules and therefore deterministically. This phenomenon can be found in all possible forms of natural systems – from physical and chemical (e.g. chemical reactions) to biological systems (e.g. the behavior of ant colonies).

A typical example of a relatively well examined system behaving according to the laws of deterministic chaos is a turbulent flow. The structure of a developed turbulent flow is characterized by vortex coherent structures, whose size is based on certain rules, but in relation to which the instantaneous position and orientation of the specific vortex in the space is random.

The known mathematical model of a flow fluid is the Navier-Stokes equation. In each case, it is a deterministic mathematical model. However, it is being shown that under certain conditions, extreme intensifying of perturbations of a certain character in a stream field can occur. Therefore, the system functions as a filter, which suppresses some perturbations and strengthens others. This process, which from the beginning is linear, leads after a certain amount of time, when perturbations are strengthened to above a certain level, to massive application of non-linearities and to the transition of the system to a chaotic state.

This is a general characteristic of dynamical systems described with a non-linear mathematical model. Whereas with linear systems, the response corresponds to the impulse, at least as far as amplitudes are concerned, with non-linear systems sensitivity varies depending on the conditions and the character of the perturbations. With extreme increases in sensitivity towards perturbations of a certain type, a problem of system stability is involved, when the quality of its behavior changes and it enters a chaotic state. Chaotic behavior is characterized by situations when very small and practically immeasurable impulses prompt major changes in the behavior of the system. If we are not capable of indicating these impulses, then behavior without
an apparent cause appears chaotic. During chaotic behavior of the system, its complexity grows (the number of active degrees of freedom).

It has been shown that linear systems are mere idealization, and in reality no real system can be perfectly described by the linear mathematical model. The linear model can apply for the actual system with sufficient accuracy only for small shifts from the normal condition, but there is a certain limit, above which the behavior of the system becomes non-linear. However, most systems are strongly non-linear. The linearized model for such systems applies only for infinitesimally small displacements. All dynamical systems occurring in nature are basically non-linear, and under certain conditions their behavior can be characterized as deterministic chaos. Therefore, chaotic behavior is so common in Nature.

The reason why the chaotic behavior of dynamical systems was for so long outside of the field of interest of scientists is that they cannot be modeled with the help of a simple linear mathematical model, which in the past – due to its simplicity – was used practically exclusively for the study of dynamical systems.

Flowing fluid is a relatively complex continuous dynamical system with great variability of boundary conditions, and it can be characterized with a very high number of degrees of freedom. The study of the properties of such a system is generally very difficult technically, and even with the resources available today, computer technology cannot manage the simulation even of very simple cases of turbulent flow from technical practice\(^6\). Therefore, we will demonstrate the chaotic behavior of dynamical systems in a much easier example using a small number of degrees of freedom. The basic mechanism of the occurrence of chaos is common to all dynamical systems, regardless of their complexity. We will choose an example that basically kicked off the era of the systematic study of chaos – Lorenz system.

Edward Lorenz served at the beginning of the 1960s at the Massachusetts Institute of Technology, where he created a simple mathematical model of the Earth atmosphere, on which he attempted to study weather, specifically forced convection in the atmosphere. For simulations, he used a digital computer, which was a top model at the time, but was very primitive from today point of view. It was a Royal McBee LGP-30 computer with 16 kB of memory, which could calculate 60 multiplications per second. Its calculations were with the accuracy of 6 valid digits. Lorenz performed rounding of the initial condition to 3 valid digits and expected that this rounding would not have a major effect on the results, but encountered unstable behavior of the mathematical model. He gradually simplified his mathematical model, which originally had 12 dimensions, to the known three-dimensional Lorenz system from 1963:

\[
\begin{align*}
\frac{dx}{dt} &= Ra(y - x), \\
\frac{dy}{dt} &= -xz + Pr x - y, \\
\frac{dz}{dt} &= xy - bz.
\end{align*}
\]

This mathematical model captures the basic properties of convective flow in the atmosphere, which is warmed by the surface from the bottom and cooled from the top. This forms rotational motion of air particles, and when the warmed particles rise, they cool and begin to drop, only to be warmed and rise again. This phenomenon is known as Rayleigh-Bénard instability (more details will be presented in the chapter about instability). The boundary conditions are somewhat idealized: flow in the upper area is considered without shear stress instead of more realistic conditions of the same velocity; in the transverse direction, the periodic boundary condition is considered instead of walls, and the entire case is modeled as a planar spatial location. A diagram of this model is shown in Figure 3.6; it is a Rayleigh-Bénard cell, which is periodically repeated in the transverse direction.

\(^6\) We have in mind the method of “direct numeric simulation”, turbulent flows are usually resolved in technical practice with the help of turbulence models. For details see Chapter 9.
Figure 3.6 – Diagram of Lorenz system – Rayleigh-Bénard cell

Variables $x$, $y$ and $z$ in equation (3.1) are not coordinates in space, and their physical importance is somewhat abstract. Variable $x$ represents the velocity of the rotation of fluid particle motion, and the positive orientation is clockwise. However, variable $y$ is the difference between the temperatures of rising and falling fluid. Variable $z$ characterizes the deviation of the vertical profile of temperature from the linear course. Parameter $Ra$ is a Rayleigh number, and $Pr$ is a Prandtl number, and finally parameter $b$ represents fluid cylinder aspect ratio during convection, meaning the ratio of its length and diameter.

From a mathematical point of view, the system of equations (3.1) has the following properties:

- The equations are autonomous, which means their right side by default does not contain the time, and coefficients are constant;
- They contain only the first time derivatives. The result of this, together with the considered autonomy of the system, is the fact that its development depends only on the instantaneous values of variables $(x, y, z)$ and not on their history;
- The equations are non-linear; see terms $xz$ and $xy$ in the second and third equation;
- The system is dissipative. This conclusion stems from the presence of diagonal parts of the equation system, which correspond to the solution being figured out;
- The solution of the system of equations is limited in the space of variables.

Lorenz performed a mathematic simulation of the system, which amounted to numeric integration of equations in time for various values of parameters and for various initial values of variables. For the values of parameter $Pr \leq 1$ the solution leads to the fixed state $x = y = z = 0$, when all convections expire.

Figure 3.7 – Lorenz strange attractor

cold wall

hot wall
The usual parameters for atmospheric conditions $Ra = 10$, $Pr = 28$ and $b = 8/3$ cause chaotic system behavior, in which the direction of the rotation randomly changes. Dynamical systems are characterized by a limit state – by attractor, which occurs after a certain transitional period depending on initial conditions. This limit final state can be displayed in the phase space either as a point – final condition of calm, to which the system is leading or as a limit cycle – a closed curve that corresponds to periodic motion. The attractor belonging to Lorenz system under certain conditions shifts away from this concept; its settlement does not occur even after a very long time, and a non-converging curve occurs. This attractor is highlighted for a limited time interval in Figure 3.7. It is the first of the strange attractors characterizing the chaotic behavior of the dynamical system, which was subjected to thorough systematic examination. It has certain unusual properties:

- It is comprised of a continuous curve in a space that generally starts at a certain initial point, but it can have an unlimited length. However, it fills a certain exactly defined sub-space in the phase space, from which it never emerges;
- It neither intersects itself nor interferes with itself nor repeats;\(^7\);
- It has the property of fractals, and its structure is similar at various scales;
- Its course in space is random, chaotic and unpredictable.
- It has been shown that the critical value of parameter $Ra$ during the values of parameters $Pr$ and $b$ above is equal to approximately 24.74; for lower values the development of the system is directed into a single point in the phase space, and for higher values we reach continuous motion with chaos elements.

The butterfly attractor consists of two branches. One is characterized with the positive value $x$ and the other with a negative value and represents the rotation of cylinders in one or another sense. Irregular jumps occur between the two branches. This attractor became a symbol of the first pioneers during the examination of chaos, and its similarity to butterfly wings inspired Lorenz in a lecture in 1972, when he spoke about the topic of weather predictability. At that time he stated with exaggeration, “The motion of butterfly wings somewhere in the Brazilian jungle can cause a tornado in Texas.”

\[ \text{Figure 3.8 – Time series of the } x \text{ variable simulating Lorenz system} \]

Figure 3.8 shows the result of simulation for various boundary conditions. Figure 3.8 (a) represents simulation of the course of variable $x$ in time $t$ carried out for initial conditions $[x(0), y(0), z(0)] = [0,1;0,1;0,1]$. Figure 3.8 (b) shows simulation for slightly changed initial conditions, variables labeled with hat $[\hat{x}(0), y(0), \hat{z}(0)]$, and their values are $[0,100001;0,1;0,1]$. The initial value $x(0)$ has been changed by one thousandth of a percent.

\(^7\) Henri Poincaré theoretically derived a theory, according to which the status of a non-linear system need not repeat after a certain amount of time. It was shown later that the period of repetition of the particular condition is in relation to non-linear systems extremely large and almost infinite.
which is a value that we are unable to distinguish during any real macroscopic experiment. It is apparent from the graph that the time course of coordinate \( x \) for both cases is practically the same up to time of approximately 30, and then both cases are developing in an entirely different manner. This is especially visible in the graph in Figure 3.8 (c), where the difference of variables \( x(t) - x'(t) \) is highlighted for both cases of initial conditions. We can see that value \( x \) jumps from positive values to negative and vice versa, and the motion along two branches of the butterfly attractor corresponds to this. The jump from one branch to the other is indeed the critical phenomenon that determines the further development of the system. It has been shown that the jump is the result of unstable behavior of the system occurring in connection with its non-linear basis, and its occurrence in a certain configuration of the system is a result of the effect of very small perturbations. These perturbations originate in the imprecisely defined parameters of the problem, their initial or boundary conditions, or in the case of mathematical simulation, in any small rounding errors in the calculation system.

Other non-linear systems behave in a qualitatively similar manner, such as turbulent flows of viscous fluid. The condition of fluid flow can from the point of view of the stability of the solution be characterized by its non-dimensional mean velocity of flow according to the Reynolds number. For Reynolds numbers higher than the critical value, chaotic behavior of flow fluid occurs, and we say that a transition from a laminar state of flow to a turbulent state is occurring. The resulting chaotic behavior and the development of such a complicated system is significantly more complex than with the simple Lorenz system. Chaos occurs in various locations at different moments in time. The resulting coherent structures then develop powerfully and mutually interact.

### 3.3.3. Process of self-organization – coherent structures

The process of self-organization of a large dynamical system can occur under the assumption that it is sufficiently far from the balanced state. Ilya Prigogin demonstrated a process that leads to the occurrence of coherent (according to Prigogin “dissipative”) structures. This process relates closely to terms such as stability, reversibility and the arrow of time. For this idea about the spontaneous transformation of “disorder” into “order” he was awarded the Nobel Prize in 1977.

The spontaneous occurrence of coherent structures during the flow of fluid was proved both experimentally and theoretically, and it was also confirmed with the help of a mathematical simulation. Figure 3.9 shows the result of direct numerical simulation of the flow of fluid, when the initial conditions for the flow field with random fluctuations were assigned. After a certain amount of time, the spontaneous creation of vortex structures occurred (the Figure shows the vortex lines). This Figure was wittily named the “box of worms”.

![Figure 3.9 – Spontaneous vortices in flow fluid (“box of worms”)](image)
The existence of highly organized and arranged structures in a turbulent flow is already a generally accepted fact today. These structures are part of turbulence itself, and moreover they are its basic building blocks. The behavior and parameters of these structures are generally unpredictable in terms of deterministic chaos, but each individually reports significant spatial coherence – organization. It is a mistake to connect unpredictability with a lack of spatial organization, because during it the existence of well-defined spatial structures cannot be expected. If we look at the instantaneous condition of a turbulent field, we can see a system of turbulent vortex structures, which are unpredictable as far as their phase is concerned (the position and orientation in space), but they maintain their geometric shape for a much longer period than the typical duration of a loss of unpredictable behavior. Under these conditions, a classical predictability approach can be applied successfully to analysis of such flow field.

The perception of turbulence as a synonym for order is no longer difficult if we understand order as existence in organized space, meaning coherent structures. Traditionally the term turbulence has often been associated with disorder and chaos, and laminar flow was a synonym for orderly motion. If we look at this problem from a macroscopic point of view, anthropomorphic scales, then actually turbulent motion can appear to be random, especially if we take into consideration the very quick and dynamic development of the flow field.

In a microscopic scale, however, the opposite is true – laminar motion appears chaotic, and turbulence is highly organized. The vortex structures contained in turbulence correspond to controlled synchronized motion of a large number of molecules. From this point of view, the transition from laminar flow to turbulent flow is a process of self-arrangement of flowing fluid.

Coherent structures play a key role in the process of mixing, stability, generating noise, etc. During a transition from the laminar condition of flow to a turbulent one, instability is characterized with a certain frequency, which can be predicted, or during artificial periodic excitation, the coherent structures can be monitored and analyzed relatively easily. Contrastingly, in developed turbulent flows, where these structures are hidden in a chaotic process, their detection is a very complicated problem. This task is practically impossible to resolve using analog methods of signal processing, and therefore systematic research of coherent structures is closely linked to the development of the digital technology of data collection and signal processing.

The identification of coherent structures in a chaotic turbulent flow field is generally a very complicated task, which requires a necessary set of experimental data as well as the application of special methods for signal processing. A large number of methods of identifying coherent structures exist. Some are very universal, and others are specialized for a specific type of coherent structures. The difficulty with identifying coherent structures is related to the fact that the definition of coherent structures itself is not sufficiently clearly and universally defined to cover all variations.

Generally the term “coherent structure” refers to an area of flow fluid that shows considerable coherence, meaning a mutual relationship between the motions of particles contained in the fluid. However, this definition is too general and cannot be used for the purposes of identifying coherent structures easily. The exact definition of coherent structures in turbulent flows has not been established even in the continuously growing community of people who have been intensively dealing with the problem of the research of these structures recently. Some definitions are focused only on vortex structures, and therefore take into consideration only “coherent vortices”. A vortex is generally one of the most stable structures in a flow fluid, and vortices are typical coherent structures. In any case, a certain level of organization is expected, which lasts for a certain period without qualitative changes. Some definitions are based on the energy principle and others on a geometric description.

If we attempt to summarize certain common characteristics of the definitions of a coherent vortex, which are specified in literature, then we can say that a coherent vortex is obviously an area in flow fluid, which:

- Contains sufficiently concentrated vorticity which is such that the paths of fluid particles create closed curves in their vicinity
- Represents an area with significant content of kinetic energy
During the “life time”, which is longer than one vortex revolution, maintains its characteristic shape

Is characterized by an unpredictable flow field.

Coherent vortices can also have a relatively regular and almost periodic character. For example, they can involve vortex structures created in connection with hydrodynamic instability in the shear layer. We can describe such vortices as pseudo periodic.

On the contrary, a coherent structure is represented by an area in flow fluid which at a particular moment in time reports a certain degree of organization due to a certain value characterizing flow (velocity, vorticity, pressure, density, temperature, etc.). This definition is much broader than with a coherent vortex. It is obvious that each coherent vortex is also a coherent structure, and the opposite claim is not valid.

Typical examples of coherent vortices are vortices created during Kelvin-Helmholtz instability, Karman vortices in the wake of a bluff body and capillary vortices occurring in the laminar boundary layer during the loss of linear stability or during the process of regeneration. On the contrary, coherent vortices do not include streaks created in a turbulent and transitional boundary layer, since these are related to the occurrence of low velocity areas and speeded up fluids and not related to the concentration of vorticity. Coherent vortices as defined above also do not include various stationary vortex structures, such as “starting” vortices created at the ends of wings, since their behavior is predictable. Another example of non-vortex coherent structures is the “bursting phenomenon”, which is a basic mechanism for generating Reynolds stresses in a turbulent boundary layer. It basically amounts to a synchronized pair of events, first the sweep of fluid from the external area towards the wall, followed by as a result of the conservation of the material the phase of ejection of slowly moving fluid in the direction away from the wall. This will be described in detail in Chapter 10.

3.4. Definition of turbulence

Despite everything that has been said above about turbulence (or because of it), its definition is difficult. We can say, for example, that turbulent flow is flow that is “irregular” in time and space. This of course is not an exact mathematical definition. The streams that we describe as “turbulent” can have very differing properties. They can be spatial or almost planar, and they can contain significantly organized and almost regular structures. A common necessary characteristic is that these streams are capable of mixing and transferring mass much faster than if only molecular transport mechanisms are applied. This property is considered the most important for its practical application. For example, in engineering applications, the coefficient of turbulent transfer of heat or turbulent resistance strength is monitored, which is related to the turbulent transfer of momentum.

What we regard as turbulence is therefore better to express as a list of properties and attributes that can help us to identify turbulent flows:

1. **Randomness**: Turbulent flow is unpredictable in the sense that small random perturbations during a particular period of time are amplified to that level, and after a certain period of time deterministic prediction of further development becomes impossible. This fact could seem to conflict with the fact that turbulent flow is describe in thorough detail by Navier-Stokes equations, which of course are of a deterministic character. The effect of their non-linear nature can under certain circumstances cause a situation in which perturbations of a certain type are very strongly amplified in time. Such perturbations could be related to the lack of precision of the assignment of initial conditions or related to Brownian motion of fluid particles, which is not modeled by the equations, because the fluid in this case is considered a continuum. The consequence of this situation is the unpredictable behavior of a specific turbulent flow. Of course, in a statistical sense, the development of turbulence can be considered predictable, since it involves deterministic chaos.

2. **Diffusivity**: Mixing of transported scalar quantities occurs relatively more quickly than during molecular diffusion. This characteristic obviously has important practical consequences, and turbulence is characterized by an increase in mixing of fluids. The intensity of this mixing can be several orders of magnitude greater than mixing occurring as a result of molecular diffusion. We can estimate that the coefficient of molecular diffusion of fluids is in technical
applications at least two orders smaller than the typical value of the coefficient of turbulent diffusion, and in the case of planetary flows (atmospheric phenomena or ocean streams), this difference can be significantly greater – typically up to 7 orders greater!

3. **Vorticity**: Turbulent flows are characterized by high local values of vorticity related to the presence of vortex structures. The field of vorticity is generally non-homogeneous and changes dynamically in time. Vortex structures tend to be referred to as coherent vortices or more generally coherent structures.

4. **Scale spectrum**: Vortex structures, which occur spontaneously in a turbulent flow field, are characterized by a wide scale of length measuring units. Their size is limited from the top by the dimensions of the shear areas in which they occurred and from the bottom by the size of vortices subject to dissipation in direct connection with the viscosity of the fluid. Thus the structures size is characterized by dense spectrum typical for fractals. Related to this is the fact that the turbulent flow field can be characterized as a dynamical system with a “very high” number of degrees of freedom.

5. **3D structure**: Vortex structures occur in the space of a turbulent flow field in random locations and with random orientation. The 3D structure of the vector field of velocity fluctuations originates from this situation. During certain boundary conditions, the structures greater than the certain limit size can be spatially arranged; for example, they can have a planar character. This relates, for example, to flows in thin layers, where the dimensions of the area enable the occurrence of vortices of larger scales than the thickness of the layer only with vorticity oriented across the layer and not in parallel with it. For smaller scales, the flow field even in these cases is spatial.

6. **Dissipation**: Turbulence is a dissipative process, which means that the kinetic energy of the motion of a fluid is dissipated at the level of small vortices and changes to heat. Therefore, in order for turbulent flows to be conserved over the long term, it is necessary to supply energy to the system from outside. This is done in the area of large scales; energy is collected from the main stream. The energy is then transferred towards smaller scales with the help of cascade transfer.

7. **Non-linearity**: Turbulent flows are basically non-linear, and their occurrence is conditioned on the application of non-linearities, when a growth of small perturbations occurs. The development as well as the interaction of individual structures in the turbulent flow field can be described only with a non-linear mathematical model.

This “definition” is limited only to listing attributes, meaning necessary properties of turbulent flows. Certain other properties are apparent from the aforementioned properties, and the other properties are contained in them along with certain conclusions regarding the size of certain non-dimensional characteristics. The characteristic of turbulence with very practical importance is the ability of mixing (related to diffusion). The variable \( l \) is the characteristic length of the energy vortices, meaning the largest vortices contained in the turbulent flow, and \( u \) is characteristic in time variable velocity. A very rough analogy between the process of mixing connected with turbulence and non-coherent random motions can enable us to define the coefficient of turbulent diffusion, which is proportional to \( l \cdot u \). The coefficients of molecular diffusion of momentum \( \nu \) (more often called molecular kinematic viscosity) and \( \theta \) (thermal diffusivity) are well-known. The ability of the increased transfer of these two physical variables (momentum and heat) in the turbulent flow is characterized by the value of non-dimensional coefficients \( l \cdot u / \nu \) and \( l \cdot u / \theta \) clearly greater than one, while in laminar flow it approaches one. The first of these parameters is called a Reynolds number, and the second is a Peclet number.

Turbulent flows are naturally unstable, and any small perturbation as a result of non-linearities of the equations of motion in time is quickly intensified. However, laminar flows behave exactly oppositely. Streamlines, which are interrupted by an obstacle, return to their original configuration. Viscous forces in laminar stream prevail. They suppress perturbations and prevent the occurrence of turbulent flow.

A lot of evidence exists both from experiments and from numerical simulations, which clearly show that turbulent flows are swirling, which means that their swirling characteristic acquires non-zero values at least in certain areas of turbulent flow. It is interesting to examine
how turbulent flows can occur in originally non-swirling flows. This process is obviously related to the effects of viscosity, because as a result of Kelvin theory, non-swirling flow is conserved in an ideal fluid, and the presence of walls and obstacles related to the condition of zero velocity on surfaces causes the production of vorticity. The production of vorticity can be further sped up by various mechanisms, such as stretching of vortex fibers, which will be described later.

3.5. Examples of turbulent flows

Turbulent motion is the most common motion in nature. Laminar flow is rather an exception and is limited to flow that can be characterized by very low velocity and thus Reynolds number (Re) values. In view of the definition of Re, this means that either the flow speeds are very low (such as melting of glaciers) or the typical dimension of the area is very small (such as motion of microorganisms in fluid) or the fluid shows extremely high viscosity (such as the motion of lubricant in bearings). Of course, a combination of these situations can also be considered.

In this chapter we will show typical cases of turbulent behavior of fluids. Some cases that cannot be clearly included into the area of fluid dynamics, but which have a large practical or theoretical importance, will also be mentioned.

3.5.1. Grid turbulence

A classic example of turbulent flow is flow behind a grid made from rods, which have regular square eyes. Behind individual rods, wakes are formed, which interact with each other and very quickly cause flow of a homogeneous structure (at a distance of about 20 spaces from the grid). The resulting flow, which is usually referred to as “grid turbulence”, has certain beneficial properties. Mainly it is to a great extent homogeneous in a statistical sense in a level parallel to the grid generator of turbulence. The fluctuations also show a high level of isotropy, and deviations are in order of percentages. If we use some of the point methods for measuring velocity (a typical hot wire anemometer), then the measured courses of velocity in time are randomly variable with probability density function, which is close to Gaussian. On the other hand, changes are occurring in the flow structure in the direction of the main flow (upright towards the level of the generator), and with increasing distance from the turbulence generator, the intensity of fluctuations of all parts of velocity declines, and the size of the largest energy vortices increases. An example of flows behind grids visualized by smoke is shown in Figure 3.10, and the flow is from left to right.

For its beneficial properties as well as for its relatively easy achievability in laboratory conditions, grid turbulence has been considered an etalon of turbulent flow.

Grid turbulence can be generated during the passing of flow fluid through a standing grid or during passage through a moving grid by originally calm fluid. These two situations are as a result of the invariance (symmetry) of equations of motion equivalent due to the Galilean transformation.
3.5.2. Free shear layers

The occurrence of free shear layers is unusually common, such as during surrounding of bodies or during flows through curved or non-prismatic (expansion) channels or at the boundary of an area of flow fluid in an unlimited space (jet). A free shear layer is nearly always unstable and results in the creation of vortex structures. In practice, we encounter free shear layers everywhere where a jet of fluid blown into a calm environment occurs or in connection with separation of a boundary layer.

![Figure 3.11 – Free jet](image)

As an illustration, we are presenting in Figure 3.11 flow in the area of a jet issuing left from a wall. A free shear layer occurs on the upper and lower boundary of the flow area. The fractal structure in the mixing area is apparent.

3.5.3. Boundary layers

During flow in the boundary layer on the surrounded surface, the decisive parameter is the Reynolds number, where the length parameter can be thickness of the layer or the distance of the particular location from the beginning of the boundary layer, meaning for example from the leading edge. At a certain value of this parameter, a transition occurs of the boundary layer to turbulence. The boundary layer also has a turbulent structure. In Figure 3.12 we can see the typical structure of a turbulent boundary layer at a relatively low Reynolds number. The wall is at the bottom, and the fluid flows from left to right. In the Figure, vortex structures are apparent inside the boundary layer, along with its irregular border. This is an instantaneous Figure, which constantly changes, but whose character remains.

![Figure 3.12 – Turbulent boundary layer](image)

3.5.4. Wakes

Wakes behind bluff bodies have a turbulent character with a dominant quasi-periodic low-frequency component. In relation to bluff bodies, the Reynolds number is decisive, where the length parameter is the transverse dimension of the bluff body. A typical situation is transverse surrounding of a cylinder, when a quasi-periodic von Kármán-Bénard vortex street occurs in the wake. Figure 3.13 shows a visualization of flow of clouds behind Juan Fernandez Island in the Pacific, where we can see both regular vortex segments in the wake and a fractal structure. The island is in the upper left corner, and the wind direction is diagonal. The centers of vortices (dark areas) are apparent, and the vortices gradually float in the wind direction.
3.5.5. Heat transfer

Also during flow combined with heat transfer, we can often observe behavior of fluid that can be described as turbulent. If fluid flow occurs as a result of heat transfer, it is referred to as natural convection. Thermal energy then causes fluid flow, which under certain conditions can be turbulent. A typical example is Rayleigh-Bénard convection, which was already discussed in the chapter about deterministic chaos.

A detailed example of flow is presented here. Figure 3.14 shows a photo of the surface of the sun, in which turbulent convective flow in the solar atmosphere is very apparent. This is caused by differences in temperatures between the surface of the sun and higher layers of its atmosphere and the lower temperatures on the surface in areas of sun spots. The photograph shows obvious turbulent sections and a cell structure in the background, which is related to Rayleigh-Bénard convection.
3.5.6. Chemical turbulence

Chemical reactions are processes with various non-linear dynamic characteristics. Non-linearities have their origin in the interaction of various particles between each other and in the behavior of individual particles. An example is a Belousov-Zhabotinsky reaction, during which an oscillating reaction occurs without any variable external influences. It has been shown that for achievement of a homogeneous structure of a mixture (reactants are citric acid, potassium bromide, sulfuric acid and cerium ions), very intensive mixing is necessary, or otherwise the result is non-homogeneity of both a stationary character (Turing structures) and a non-stationary character. Through intensive mixing, a structure can be maintained more or less homogeneous. Of course if we do not apply mixing, certain unstable frequencies occur in flows, which can culminate into quasi-stationary structures, which differ from each other based on their chemical contents. In Figure 3.15 we can see the instantaneous condition during a Belousov-Zhabotinsky reaction, when regular spiral structures are created as a result of periodic oscillations, which are related to global Hopf bifurcation. The chemical particles participating in the reaction differ by color. This is a very stable process also known as a “chemical clock”. If we breach the equilibrium of the particles entering into the reaction, then the reaction will either be stopped or will transit to a stormy turbulent regime.

![Figure 3.15 – Belousov-Zhabotinsky reaction](image)

It is interesting that similar structures can also be observed even among entirely different processes, such as wave formation on the surface of a liquid or during the growth of a colony of single-cell organisms.

3.5.7. Burning

Burning is another area with the occurrence of a whole range of turbulent conditions. It basically involves a combination of the two previously mentioned cases; it is a chemical reaction which is strongly exothermal and under normal circumstances irreversible.
Figure 3.16 – Fractal structure of a flame

Figure 3.16 shows a photograph of burning of a mixture of natural gas and air. It is blown from the left. The dark areas represent an unburned mixture. The bright line represents the area of the flame. We can see the flame boundary in the typical V shape, and the structure of the burning surface is fractal. In the Figure, there is also an even cut, and the surface is in fact spacious.
Basic equation of fluid dynamics

In this chapter we will formulate the basic ideas and approaches during a mathematic description of flow fluid.

3.6. Basic assumptions

A fluid, like every real substance, is comprised of molecules. When considering the behavior of real fluids, an anthropomorphic approach is applied, when a random object is considered from the point of view of a person as a basic scale for measuring all objects. From this point of view, it does not make sense to examine phenomena whose measuring scales are not capable of capturing human senses, under the assumption that the micro structure of the substance is adequately captured using a physical model. The behavior of the micro structure of fluids in molecular scales is examined to determine the statistical mechanics of fluid. For resolving tasks related to the mechanics of fluid from the real world, however, a different approach is used, one in which the fluid is regarded as a continuum and its molecular structure is ignored. The justification of adoption of this physical model needs to be verified carefully, as it depends on the physical properties of the particular fluid.

Let us consider as a working fluid air under “normal” conditions, i.e. pressure equal to the average atmospheric pressure (10^5 Pa) and “room” temperature (20° C). Then the average distance between the molecules of gas comprising air is approximately 3.10^-9 m, the mean free path of molecule \( \lambda \) is approximately 6.10^-8 m, and the average time period between two collisions of molecules is approximately 10^-10 s. For comparison, the smallest length scale \( l \) in “anthropomorphic” flows of fluid, meaning in fluids flow in machines or in an environment inhabited by people, is usually greater than 10^-4 m, and during the mean velocity of flow of 100 m/s the time scales are greater than 10^-6 s. Therefore, even in this somewhat extreme situation, the scales of flow exceed molecular scales more than three times. This means that the number of particles in the smallest structure occurring in the space in flow fluid is \((10^4)^3 = 10^9\).

The acceptability of the continuum hypothesis is often verified with the help of the Knudsen number, which expresses the relationship between the mean free path of molecules and the smallest scale:

\[
Kn = \frac{\lambda}{l} \tag{4.1}
\]

An environment is considered a continuum if there is fulfillment of the condition \( Kn \ll 1 \); in the aforementioned case \( Kn \leq 10^-3 \) this condition is obviously fulfilled.

With a sufficiently small Knudsen number, the scales of molecular motions and macroscopic motions of fluid are separated. Then a certain “elementary” volume can be taken into consideration, which represents a material point from the point of view of motion of fluid as a continuum, but also contains a sufficient number of molecules for it to be possible to express individual physical quantities such as the density \( \rho \) and velocity \( \mathbf{u} \), which we can then consider a continuous function of time \( t \) and position \( \mathbf{x} \) in space \( \rho(\mathbf{x},t) \) and \( \mathbf{u}(\mathbf{x},t) \). However, considerations about the behavior of individual molecules lose their significance. We can then talk about physical quantities in an infinitesimal point, and it is also possible to define derivative and gradients of these quantities in space. If we consider fluid as a medium with a fractal structure, then this structure is limited by the sizes of the volumes that have been defined above. This expectation is necessary, because otherwise it would not be possible to define derivative in these structures.

In this work we will consider only fluids whose physical characteristics can be considered continuous. We will understand particles of fluid as an area in the fluid with a typical dimension of 10^-4 m, which has the properties of continuum in the spirit of the definition above.
3.7. Euler and Lagrange description

The motion of fluid and its kinematics can be studied using two methods. The first method involves selecting from the volume of fluid a random elementary particle based on the hypothesis of continuous fluid and monitoring of its motion. The second method involves monitoring of the changes in kinematic quantities in individual points of the flow area in space. In the first method, which is called the Lagrange method, we investigate the motion of fluid from the point of view of individual particles, while in the second method, the Euler method, we examine the field of kinematic quantities directly. Let us introduce both methods in greater detail.

When using the Lagrange method, we select in initial time \( t_0 \) a particle specified by position vector \( x_0 \). We can describe the position of the particle in the following moments using the equation:

\[
x = x(x_0, t).
\]  

The independent variable values in this equation are referred to as Lagrange variables. Since a continuous environment is involved, this function must be a continuous time function, and continuity in view of the position in space is not necessary. The velocity of \( u \) and acceleration \( a \) of a particle can be expressed using a simple derivative with respect to time:

\[
u = \frac{\partial x}{\partial t}, \quad a = \frac{\partial^2 x}{\partial t^2}.
\]  

During application of the Euler method, we examine the state of the flow fluid in a particular point of space \( x \). The term “Euler variables” is used for the vector of the position of the examined point and time. The kinematic state of the fluid in the examined fixed point is characterized using the velocity vector \( u(x, t) \). We now express the acceleration of the particle, which in the particular moment in time takes over the examined point in space. For time interval \( dt \) there is a change of its coordinate \( x \) by \( dx \). Then for the \( i \) part of the velocity vector \( u_i + du_i \) in time \( t + dt \) we can write the following Taylor expansion, in which we have ignored the terms of higher orders:

\[
u_i(x_1, x_2, x_3, t) + du_i = u_i(x_1, x_2, x_3, t) + \frac{\partial u_i}{\partial x_1} dx_1 + \frac{\partial u_i}{\partial x_2} dx_2 + \frac{\partial u_i}{\partial x_3} dx_3 + \frac{\partial u_i}{\partial t} dt.
\]  

When using the Hamiltonian operator nabla we can write a vector equation:

\[
u(x, t) + du = u(x + dx, t + dt) = u(x, t) + (dx \cdot \nabla)u + \frac{\partial u}{\partial t} dt.
\]  

For acceleration \( a \) we then get the following result, where operator \( D/Dt \) shows derivative with respect to time in the Lagrange sense:

\[
a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u.
\]  

We can generalize this approach and apply it to a random vector function of Euler variables \( f(x, t) \), which we want to derive based on time monitoring the motion of the particle:

\[
\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (u \cdot \nabla) f.
\]  

We describe the total Lagrange derivative \( Df/Dt \) as a “substantial” or “individual” derivative of function \( f \) based on time. The first term on the right side \( \frac{\partial f}{\partial t} \) is a “local” derivative, and the second term \( (u \cdot \nabla) f \) is a “convective” or “flow” derivative of function \( f \) with respect to time. This term is often also referred to as “carrying”, “convective” or “advective”.

The same method can also be applied to the scalar function of coordinates and time.
In practice we usually work with the help of the Euler method, when we directly resolve the field of monitored quantities in the examined area. The Lagrange method is used in various special tasks, such as examination of the range of the scalar quantity bound to the fluid particles.

The term “trajectory” is directly connected to the Lagrange method of describing continuum behavior. Trajectories, or path lines, are described directly by the parametric vector equation (4.2) and/or (4.3) and define the path of particles in relation to areas in time. We can see trajectories in an experiment with the help of visual methods. This situation is shown in Figure 4.1.

\[
\begin{bmatrix} x_1(t), x_2(t) \\ x_1(t_0), x_2(t_0) \end{bmatrix}
\]

**Figure 4.1 – Lagrange description**

Euler description method is closely related to the term “streamline”. A streamline is defined as an aggregate of points in which the vectors of velocity of the fluid in a particular moment in time are tangent. We can express a parametric equation of a streamline as follows:

\[
\frac{dx_1}{u_1(x,t)} = \frac{dx_2}{u_2(x,t)} = \frac{dx_3}{u_3(x,t)}, \tag{4.8}
\]

where \( x_i \) and \( u_i \) are parts of the vector of the position and/or velocity in the particular moment, and time \( t \) is a constant parameter. It can be shown that if the functions do not simultaneously acquire zero values and they are unambiguous and continuous, including first derivatives with respect to coordinates, then each point of the vector field is passed through by only one streamline. Euler method of describing the flow field is shown in Figure 4.2.

\[
\begin{bmatrix} x_1(t), x_2(t) \end{bmatrix}
\]

**Figure 4.2 – Euler description**

A complete system of trajectories of all fluid particles is a perfect description of kinematic behavior of fluid in a particular time interval. An equivalent complete description is provided by a system of streamlines for all area points.

Streamlines represent an Figure of flow in a certain moment in time, while trajectories characterize the motion of an examined particle in a time interval. Generally the velocity field is a function of time, and therefore the Figure of flow changes. Then streamlines and trajectories are obviously presented using mutually differing curves. Notice that in the case of a stationary or developed flow, which does not change in time and in relation to which local acceleration of the velocity field is zero, both line systems, meaning trajectories and streamlines, combine together and are identical.
The parts of velocity then are not explicit functions of time, the streamline does not change and the fluid particle gradually flows through all points of one and the same streamline. Unfortunately, a basic attribute of turbulent flows is that it is not stationary, and this means that trajectories and streamlines in a turbulent flow field always differ.

Both presented views of flowing fluid, Lagrange and Euler, describe the entire area of fluid flow, and a clear link exists between them. Lagrange fluid coordinate can be identified with the integration constants of Euler description of trajectories.

3.8. Conservation laws

Based on the existence of symmetry, laws of conservation of various quantities exist for dynamical systems (symmetry is described in greater detail in this section). Laws of conservation of mass and momentum play a key role in the dynamics of fluids, and conservation of energy plays a role in thermodynamics.

3.8.1. Continuity equation

The continuity equation represents the application of the law of conservation of mass for the area of fluid mechanics.

Let us further consider an elementary particle of fluid in the volume of \( \delta V \) and mass \( \delta m = \rho \delta V \). The molecular diffusion of material across the boundary of an elementary particle within macroscopic scales is zero, and therefore the mass of the particle can be considered constant. This means that its total Lagrange derivative with respect to time is zero:

\[
\frac{1}{\rho} \frac{D \rho}{Dt} + \frac{1}{\delta V} \frac{D \delta V}{Dt} = 0 .
\]

It can be easily shown that the divergence of velocity can be expressed in the following manner:

\[
\nabla \cdot u = \frac{1}{\delta V} \frac{D \delta V}{Dt} .
\]

and equation (4.9), which represents a condition for continuity, can be written as follows:

\[
\frac{1}{\rho} \frac{D \rho}{Dt} + \nabla \cdot u = 0 .
\]

In the case of flow of incompressible fluid, valid equation (4.11) is generally reduced to two separate equations:

\[
\nabla \cdot u = 0, \quad \frac{D \rho}{Dt} = 0 .
\]

The first equation expresses the property of the vector field of velocity, which must report zero divergence. We can refer to the vector field of velocity as solenoidal.

All considerations in these scripts are related to incompressible fluids.

Let us notice that the condition of incompressibility does not mean a priori that the density is constant in the entire area. Therefore, the conclusions can also be applied to a non-homogeneous stratified fluid, such as sea water with variable salt content or the Earth thermally non-homogeneous atmosphere.

3.8.2. Conserving momentum

Let us examine the behavior of an elementary particle of fluid from the point of view of Newton second motion law – the law of force.

The condition of a fluid particle is characterized by acceleration of the particle via surface and volume forces affecting the particle. Acceleration of a particle needs to be considered in the Lagrange sense \( Du/Dt \). Surface forces have their origin in molecular motion and can be described with the help of shear stress tensor \( \tau_{ij}(x,t) \), which is symmetric; this means that the following applies: \( \tau_{ij} = \tau_{ji} \). Volume force is typically a gravitational force, and in special cases other volume forces can have an effect, such as centrifugal forces. We can characterize
gravitational force based on gravitational potential $\Psi$, and the gravitational force affecting a mass unit is

$$g = -ge = -\nabla \Psi.$$  \hfill (4.13)

For constant gravitational fields, it applies that $\Psi = gx$, where $g$ represents gravitational acceleration, $x$ is a coordinate in a vertical direction, and $e$ is a unit vector in this direction.

We can now express the equilibrium of the particle in the direction of coordinate $x_i$ using the

$$\rho \frac{Du}{Dt} = \frac{\partial \tau_{ik}}{\partial x_k} - \rho \frac{\partial \Psi}{\partial x_i}.$$  \hfill (4.14)

Let us now consider a special case, in which for the shear stress in the fluid Newton law applies, meaning that the fluid is a Newton fluid. If we have the burden of the element characterized using the tensor of stress $\tau_{ik}$, then stress $\tau_{s_i}$ on the general elementary surface, which is characterized by the normal vector $n_k$ can be expressed as follows:

$$\tau_{s_i} = \tau_{ik} n_k.$$  \hfill (4.15)

The total force affecting the particular surface $S$ from volume $V$ can then be expressed using the Green theorem

$$\int_S \tau_{s_i} dS = \int_S \tau_{ik} n_k dS = \int_V \frac{\partial \tau_{ik}}{\partial x_k} dV.$$  \hfill (4.16)

Then the shear stress tensor for incompressible Newton fluid will be

$$\tau_{ij} = -P \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$  \hfill (4.17)

where $P$ is pressure and $\mu$ is the coefficient of dynamic viscosity. Assuming that the continuity equation applies, for incompressible fluid the field of velocity is solenoidal, and expression (4.17) represents the tensor of stress divided into an isotropic part $(-P \delta_{ij})$ and a deviator.

If we add an expression for the tensor of shear stress in equation (4.14), we reach the Navier-Stokes equation (in the further N-S equation) for the component in direction $x_i$ in the basic form

$$\rho \frac{Du_i}{Dt} = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial P}{\partial x_i} - \rho \frac{\partial \Psi}{\partial x_i}.$$  \hfill (4.18)

We must use the N-S equation for all three parts to connect the continuity equation, and we further expect that $\rho$ and $\mu$ are constant.

We can further modify the N-S equation. Modified pressure $p$ is implemented:

$$p = P + \rho \Psi.$$  \hfill (4.19)

When using the vector formulation, we get

$$\frac{Du}{Dt} = -\frac{1}{\rho} \nabla p + v \nabla^2 u,$$  \hfill (4.20)

where $v = \mu / \rho$ is the coefficient of kinematic viscosity.

The N-S equations represent a system of non-linear partial differential equations for unknown vector of velocity $u$ and pressure $p$, meaning 4 unknown scalar quantities. We have available 3 N-S equations and the continuity equation $\nabla \cdot u = 0$. In order to solve them, we need initial and boundary conditions. If a non-moving wall appears in a stream of fluid, then the condition of impermeability of the wall applies on its surface:

$$n \cdot u = 0,$$  \hfill (4.21)

where $n$ is the normal vector in relation to the wall in the particular location. The no-slip condition also applies, which states that the component of the velocity that is tangent in relation to the surface has a zero value on the wall:

$$u - n (u \cdot n) = 0.$$  \hfill (4.22)

These two conditions (4.21) and (4.22) can be combined into one:
\[ u = 0. \]  

(4.23)

In certain specific situations it may be justified to consider an ideal inviscid fluid. The tensor of shear flow then contains only an isotropic part:

\[ \tau_{ij} = -P \delta_{ij}. \]  

(4.24)

The equilibrium of momentum then has the formulation of Euler equations:

\[ \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p. \]  

(4.25)

Euler equations, unlike N-S equations, do not contain a second derivative of velocity, and therefore they also require a different formulation of boundary conditions. For example, on a non-moving wall only the condition of wall impermeability (4.21) can be used, not the no-slip condition (4.22). Here it is necessary to emphasize that the solution of Euler equations is generally not identical to the solution of N-S equations for situation \( v \to 0 \).

### 3.8.3. Navier-Stokes equations

Navier-Stokes equations are basic equations used in fluid dynamics. Navier came up with these equations in 1823. N-S equations in themselves probably contain all aspects of the actual behavior of fluids, including turbulence, or at least experimentally verified situations indicate this so far. Nonetheless, it is still necessary to view N-S equations as a mathematical model and to confront them with experimental data. Let us also keep in mind that the hypothesis about the validity of N-S equations is based on the a priori validity of other hypotheses, such as the hypothesis of fluid continuity or the hypothesis that a fluid is a Newtonian fluid. If the adoption of these hypotheses is not justified in this particular situation, then it is very unlikely that even a mathematical model based on N-S equations can be applied successfully.

N-S equation in a component formulation:

\[ \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}. \]  

(4.26)

The physical importance of individual terms of a N-S equation is defined as follows:

- **I** variability of the flow field in time,
- **II** characterizes convection,
- **III** pressure gradient,
- **IV** effect of viscosity.

N-S equations (4.26) are specified in a component formulation, but we can also write them in a vector formulation:

\[ \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}. \]  

(4.27)

The left side of N-S equations represents substantial derivative of velocity with respect to time, while we can express the right side more easily with the help of a shear stress tensor \( \sigma_y \):

\[ \sigma_y = -P \delta_y + d_y, \]  

(4.28)

where \( d_y \) is a deviator of a tensor of shear stress

\[ d_y = 2 \mu s_y, \]  

(4.29)

\( \mu \) is a coefficient of dynamic viscosity and \( s_y \) is the strain rate tensor.

\[ s_y = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]

Then an N-S equation has the simple formulation

\[ \frac{D\mathbf{u}}{Dt} = \frac{\partial \sigma_{ik}}{\partial x_k}. \]  

(4.30)

This equation is often also referred to as a Cauchy equation and represents the general formulation of an equation describing the motion of a continuous medium.
Sometimes it is desirable to express N-S in non-dimensional coordinates. So $L$ is a characteristic dimension of the flow area, and $V$ is characteristic velocity. Let us introduce non-dimensional coordinates

$$X_i = \frac{x_i}{L}, \quad U_i = \frac{u_i}{V}, \quad P = \frac{p}{\rho V^2}, \quad \tau = \frac{\nu t}{LV}.\quad (4.31)$$

Then we can rewrite N-S equations in component formulation in non-dimensional coordinates

$$\frac{\partial U_i}{\partial \tau} + \text{Re} \left( U_i \frac{\partial U_i}{\partial X_k} + \frac{\partial P}{\partial X_i} \right) = \frac{\partial^2 U_i}{\partial X_k \partial X_k},\quad (4.32)$$

where $\text{Re}$ is a non-dimensional Reynolds number – a parameter determining the quality of a flow:

$$\text{Re} = \frac{LV}{\nu}.\quad (4.33)$$

We can see that if we do not consider the external volume forces affecting the fluid, the fluid behavior can be fully characterized by a single parameter – the Reynolds number.

### 3.8.3.1. Properties of N-S equations

N-S equations are partial non-linear differential equations of the second order. Despite their formal simplicity, mathematical theory tells us only very little about their properties. Not only do we not know a general solution of these equations in an analytical form, but to date no evidence has been presented of the most basic properties of this solution, such as its existence, smoothness and uniqueness and/or its stability in a general spatial case. An example of expression of the seriousness of this problem is the activity of the Clay Mathematic institute (CMI) in Cambridge, Massachusetts, USA. The institute at its annual meeting in Paris in 2000 announced 7 mathematical problems for the 3rd millennium and announced a reward of $1 million for the solution of each of them. One of these problems is the evidence of the basic properties of N-S equations.

A key characteristic of N-S equations is their non-linearity, which is the source of all difficulties related to these equations. Related to them are properties of the solution, which can be characterized as fractal and include phenomena such as deterministic chaos and self-organization as well as the creation of coherent structures. Another property is the non-localness of N-S equations and basically involves integro-differential equations for the field of velocity, and this field is non-local. We can distinguish two aspects of this non-localness:

- **Dynamic non-localness** – pressure in the point is defined with the help of the entire velocity field. Pressure has a non-Lagrangian character and the non-localness of turbulence in time (“memory”) is related to it. During elimination of pressure (equations for vorticity), non-localness establishes a definition of vorticity, and there is a two-sided link between the velocity field and the vorticity field (vorticity does not behave as a passive scalar).
- **Reynolds decomposition** (for details see chapter 5) – a mutual connection exists between the fields of mean velocity and fluctuations, which is not localized in time and space and has functional character. Fluctuation in the particular point and time are functions of the mean field in the entire space and vice versa.

The result of these facts is that N-S equations are not integrable, and no analytical solution exists in a closed form. One of the consequences can be chaotic behavior of the solution.

Another important property of N-S equations is their symmetry. We will discuss this property in greater detail in the following paragraph.

### 3.8.3.2. Symmetry of N-S equations

Under the term “natural law” we usually understand a set of rules that tell us how things change in space and time. It serves for predicting the future behavior of something, under the assumption that the initial condition is known. Such laws of change can be transformed to entirely equivalent statements or laws regarding invariance, meaning the invariance of a certain structure or property during any permitted change of the state of a monitored system. It has been shown that invariance leads to conserving a certain quantity in time.
The classification of all types of changes in relation to various types of invariance is dealt with by the field of mathematics – group theory. We understand a group as a set of changes that are identified by three properties: they must include the possibility that no change will occur, and they must contain the option of cancelling each change or returning it to its original state, and each two consecutive changes must yield a result that would enable us to achieve the single change belonging to the particular set.

Each of the physical laws of conservation that we know is based on a certain invariance, and this means that a set of changes forming a group of symmetry exists, which leaves these laws without changes and leads to the law of conservation. For example, conserving energy is an equivalent invariance of laws of motion in relation to shifts in time forward or backward; in other words, the result of an experiment does not depend on the moment when it was carried out, if all other conditions are identical. Conservation of momentum is equivalent to the invariance of laws of motion due to the position of the laboratory in space, and conservation of the moment of momentum is equivalent to invariance in view of the direction orientation of the laboratory. Other conserving quantities in physics, which relate to integration constants of the law of change, are proving to be equivalent other less apparent laws of nature.

$G$ describes a group of transformations affecting space and time functions $u(x,t)$, which are spatially periodic and whose divergence is zero. $G$ is considered a group of symmetry of N-S equations, if it applies that for all $u$ that are solutions for N-S equations and all $g \in G$ the $gu$ functions are also a solution for N-S equations. We are also presenting a list of so far discovered symmetries of N-S equations.

Currently, six symmetries of N-S equations are known. Assuming that the condition of the fluid system is characterized by time $t$, the vector field of position $x$ and the vector field of velocity $u$, we can characterize individual symmetries as follows:

1. **Motion in space:**
   \[ g_r: \quad t, x, u \mapsto t, x + r, u, \]
   where vector $r \in \mathbb{R}^3$ represents a vector of a shift in space.

2. **Shift in time:**
   \[ g_{\tau}: \quad t, x, u \mapsto t + \tau, x, u, \]
   $\tau \in \mathbb{R}$ is a shift in time.

3. **Galilean transformation:**
   \[ g_U: \quad t, x, u \mapsto t, x + Ut, u + U, \]
   $U \in \mathbb{R}$ is a carrying velocity and involves an inertial system.

4. **Mirroring (parity):**
   \[ g_p: \quad t, x, u \mapsto t, -x, -u, \]
   This involves “reversibility” of flows. This symmetry applies only under the assumption that there is a negligible non-linear (convective) term and generally does not apply in turbulence.

5. **Rotation:**
   \[ g_A: \quad t, x, u \mapsto t, Ax, Au, \]
   $A \in SO(\mathbb{R}^3)$ is a transformation matrix of rotation in space. However, continual rotation is not consistent with periodic boundary conditions, and only certain discrete values of rotation are permitted. Continuous rotation is permitted only for an unlimited area.

6. **Scaling:**
   \[ g_h: \quad t, x, u \mapsto \lambda^{-1} t, \lambda x, \lambda^h u, \]
   $\lambda \in \mathbb{R}$ is a scaling parameter and $h \in \mathbb{R}$ is an exponent. For an arbitrary $h$ this symmetry is valid only for inviscid fluid, for fluid with viscosity, it is necessary for it to be $h = -1$.

Symmetries are basic properties of the dynamical system, which must be inherently contained in the mathematical model of this system. If the used mathematical model reports different properties in relation to symmetries, then it is necessary to seriously consider the justification of the use of such a system and/or to carefully consider limiting the model stemming from this fact.

### 3.8.3.3. Equations for pressure

In our considerations from the point of view of the dynamics of fluids, we think of pressure from a somewhat different point of view than for example thermodynamics, where we are used to
combining the instantaneous pressure with the state quantities of gas – temperature and density. In fluid dynamics the pressure field is connected with the velocity field.

We express the divergence of the N-S equation by multiplying using the operator nabla \( \nabla \)

\[
\left( \frac{D}{Dt} - \nu \nabla^2 \right) \nabla \cdot \mathbf{u} = -\frac{1}{\rho} \nabla^2 p - \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j}. \tag{4.34}
\]

However, the velocity field is solenoidal, and therefore with consideration for (4.12) the left side of equation (4.34) equals zero. The right side must also equal zero, and therefore the following applies:

\[
\nabla^2 p = -\rho \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j}, \tag{4.35}
\]

which is Poisson equation for pressure. Its fulfillment is a necessary and sufficient condition for the solenoidal velocity field to remain solenoidal. For solution of this equation, we need boundary conditions. On the solid wall, we apply Neumann boundary condition in the following formulation:

\[
\frac{\partial p}{\partial n} = \mu \frac{\partial^2 u_n}{\partial n^2}, \tag{4.36}
\]

where \( n \) is the distance in the normal direction to the surface, and \( u_n \) is a velocity component perpendicular to the wall.

We should notice that Poisson equation expresses the non-localness of the problem. The pressure in the random point according to this equation is a function of distribution of velocity in the entire area. Pressure represents a very interesting quantity, which is a suitable diagnostic quantity for the entire area of flow.

For solving Poisson equation, we can use the Green function, and the result is a field of pressure distribution in the area of flow.

### 3.8.3.4. Formulation for vorticity field

The basic property of the turbulent flow field is its vorticity. “Vorticity” or “vortex of velocity” is defined as curl of the velocity

\[
\omega = \nabla \times \mathbf{u} = \text{rot} \mathbf{u}. \tag{4.37}
\]

Vorticity acquires non-zero values in a turbulent field. The vorticity modulus numerically equals double the velocity of rotation of the element of fluid in the particular point. Vorticity identically fulfills the continuity equation (see addendum – tensor calculus (12.10)).

We rewrite the N-S equation in the “rotation formulation” using vector identity (see also (12.11))

\[
\nabla \mathbf{u}^2/2 = (\mathbf{u} \cdot \nabla)\mathbf{u} + [\mathbf{u} \times \text{rot} \mathbf{u}]. \tag{4.38}
\]

N-S equation (4.27) is then

\[
\frac{\partial \mathbf{u}}{\partial t} - [\mathbf{u} \times \omega] + \nabla \left( \frac{p}{\rho} + \frac{\mathbf{u}^2}{2} \right) = \nu \nabla^2 \mathbf{u}. \tag{4.39}
\]

We can obtain the equation for vorticity through vector multiplication of the N-S equation using the nabla operator from the left, and after adjustment we reach

\[
\frac{D \omega}{Dt} = \frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = \nu \nabla^2 \omega + (\omega \cdot \nabla) \mathbf{u}. \tag{4.40}
\]

The term containing pressure \(-\nabla \times \nabla \rho / \rho\) for fluid with constant density has a zero value, and this is the result of vector identity (12.8). Therefore, pressure fell from the equation for vorticity.

We should notice that the equation for vorticity also contains terms comprising velocity, both on the left side (in the convective term contained in the substantial derivative) and on the right side. The physical significance of the first term on the right side is viscous diffusion of vorticity, and the significance of the second is generation or production of vorticity as a result of the non-homogeneity of the velocity field.

Equation (4.40) is usually referred to as the Helmholtz equation. The solution of this equation requires closing of the system by adding a relationship between velocity and vorticity.
One of the options is expressing velocity as an integral of the vorticity field. This gives us a system of non-linear integro-differential equations. The integral property of these equations reflects the non-local character of N-S equations. However, Helmholtz equations are much more complicated than the original N-S equations, but nonetheless this approach is used oftentimes, particularly in the dynamics of inviscid fluids. An advantage of this formulation is the fact that vorticity is often concentrated in limited areas (vortex cores), and elsewhere it is practically zero. Another advantage is the absence of pressure.

For inviscid fluid, the term with kinematic viscosity vanishes, and we reach an expression of Euler equations with vorticity:

\[
\frac{D\omega}{Dt} = \omega \cdot \nabla u.
\] (4.41)

This equation sufficiently precisely describes the behavior of large vortex structures in limited time intervals. The vortex structures often have a planar character, and this means that particularly \( u_3 = 0 = \omega_1 = \omega_2 \), and only the velocity components \( u_1 \) and \( u_2 \) and vorticity component \( \omega_3 \) have nonzero values. From this, it is apparent that for this flow the term on the right side of the equation (4.41) identically equals zero, and the following applies:

\[
\frac{D\omega}{Dt} = 0,
\] (4.42)

where \( \omega = \omega_3 \).

This result can be interpreted with the help of the following statement: “During planar flow of ideal fluid in a potential force field, the vorticity of all individual fluid particles is conserved.”

For steady flow the equation (4.42) takes on an even simpler form:

\[
(u \cdot \nabla) \omega = 0,
\] (4.43)

This result can be interpreted with the help of the following statement: “During established planar flow of ideal fluid in a potential force field, the vorticity is maintained along all streamlines.” The result of the validity of this statement is the fact that planar flow around smoothly surrounded bodies (such as aviation profiles) always lacks vorticity.
4. Turbulent flow equation

Now we will show the method of describing a turbulent flow field with the help of a statistical approach based on probability, which is currently used most often to solve engineering problems.

This approach is based on the term ensemble average set of data (a more exact definition is contained in the addendum, Chapter 12.4.1.). With its help it is possible to study the laws of probability of the occurrence of various conditions, which are described with the help of statistical characteristics. This principle was first used in turbulence by Gablelein in 1935, and Kolmogorov theory of isotropic turbulence from 1941 and many other theories are based on it.

4.1. Reynolds equations

The instantaneous states of the flow field of an incompressible fluid are described completely using a set of N-S equations complemented by a continuity equation. Solutions to these instantaneous states are difficult to apply for practical situations for various reasons, and therefore an adjustment of the mathematical model for calculation of statistically mean states is used. Already in 1894, Reynolds formulated the equations, which are named after him.

Reynolds assumes that the velocity field \( u(x,t) \) can be distributed in the following manner:

\[
u(x,t) = \overline{u(x,t)} + u'(x,t).
\]

This is “Reynolds decomposition” in time mean component \( \overline{u(x,t)} \) and fluctuations \( u'(x,t) \). Details regarding the method of averaging are found in Chapter 12.4.1. For fluctuation velocity the following obviously applies:

\[
u'(x,t) = 0.
\]

We can also apply the same decomposition to other quantities, namely pressure

\[
p = \overline{p} + p'.
\]

For more about averaging, see the addendum to this script, Chapter 12.4.

It is apparent from the fact that the field of instantaneous velocity is solenoidal that both the field of mean velocity and the field of fluctuation are also solenoidal. Therefore, the continuity equation does not apply only for fields of instantaneous velocity, but also for fields of mean velocity and for fields of fluctuation components

\[
\frac{\partial \overline{u_k}}{\partial x_k} = 0; \quad \frac{\partial \overline{u_k}}{\partial x_k} = 0; \quad \frac{\partial u'_k}{\partial x_k} = 0.
\]

We can also express \( u'_k \frac{\partial u}{\partial x_k} \) with the help of the by parts derivation

\[
u'_k \frac{\partial u}{\partial x_k} + u_k \frac{\partial u}{\partial x_k} = \frac{\partial}{\partial x_k} u_k u_k.
\]

However, the second term on the left side of equation (5.5) is according to (5.4) equal to zero, and therefore the following applies:

\[
u'_k \frac{\partial u}{\partial x_k} = \frac{\partial}{\partial x_k} u_k u_k.
\]

“Reynolds equations” originate from N-S equations through application of the operation of averaging. Let us consider N-S equations in a component formulation. First we express the formula for the mean value of substantial derivative of velocity

\[
\]

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\[
\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = \frac{\partial}{\partial t} \left( u_i + u'_i \right) = \frac{\partial}{\partial t} \left( u_i + u'_i \left( u_k + u'_k \right) \right) = \frac{\partial (u_i + u'_i)}{\partial t} + \frac{\partial \left( u_i + u'_i \right)}{\partial x_k} = \left( u_i + u'_i \right) + \frac{\partial}{\partial x_k} \left( u_i + u'_i \right)
\]

(5.7)

We can see that the mean substantial derivative of instantaneous velocity is equal to the substantial derivative of mean velocity increased by the additional term \( \frac{\partial u'_i}{\partial x_k} \). After entering the N-S equations we reach the Reynolds equation

\[
\frac{Du_i}{Dt} = \nu \frac{\partial^2 u_i}{\partial x_k^2} - \frac{\partial u'_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i},
\]

(5.8)

We can rewrite the right side of this equation as follows:

\[
\frac{Du_i}{Dt} = \frac{1}{\rho} \frac{\partial}{\partial x_i} \left[ -p \delta_{ik} + \mu \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) - \rho u'_i \right],
\]

(5.9)

and then in brackets on the right side we have the sum of the three terms, which we can interpret as stress. The first term \(-p \delta_{ik}\) represents stress caused by mean pressure, \(\mu \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)\) is then the tensor of mean viscous stress, and finally \(-\rho u'_i\) is the tensor of stress whose creation is related to velocity fluctuations. We refer to this quantity as a “Reynolds stress tensor”. In fully developed turbulent flow, it applies that the absolute size of the Reynolds stress tensor is at least by two orders of magnitude larger than the tensor of mean viscous stress. This does not apply only in thin shear areas, where the effect of viscosity is decisive. This is a case of a viscous sublayer, which is located in the in direct proximity to the wall.

In the same manner, through the application of the averaging operation, we can adjust the Poisson equation

\[
-\frac{1}{\rho} \nabla^2 p = \frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_k} = \frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_k} + \frac{\partial^2 u'_i}{\partial x_k \partial x_i}.
\]

(5.10)

We would reach the same result by calculating the divergence of Reynolds equation (5.8).

A statistical description of the flow field is therefore presented with the help of 3 Reynolds equations and a continuity equation or a Poisson equation. Therefore, we have 4 equations available. These equations contain besides 4 basic unknowns, which are components of the mean vector of velocity and mean pressure, also a Reynolds stress tensor, which figures as another 6 unknown quantities, because it involves a 2nd order symmetrical tensor. The system of Reynolds equations is therefore insufficiently determined, and we can describe it as “unclosed”. For its unique solution, we need additional information regarding the Reynolds stress tensor.

**4.1.1. Reynolds stress**

The formulation of Reynolds equations is formally the same as that of N-S equations, except for an additional term containing Reynolds stress, \(-\rho u'_i u'_j\), which plays a key role in Reynolds equations.

The physical interpretation of Reynolds stress is the average flow of momentum in the direction “i” related to a fluctuation motion in the direction “j” and/or on the contrary a flow in the direction “j” caused by fluctuation in the direction “i”. Reynolds stress therefore represents force on the surface unit related to the transfer of momentum via a fluctuation velocity field, unlike viscous stress, which represents force related to the transport of momentum in a molecular scale.
The Reynolds stress tensor is a symmetrical semi-definite $2^{nd}$ order tensor, and through a simple relationship it is linked to the correlation tensor $u'_iu'_j$. Let us further examine the properties of the correlation tensor $u'_iu'_j$.

Diagonal elements $u'_iu'_k = u'^2_k$ relate to normal stress while non-diagonal elements $u'_iu'_j, \ i \neq j$ characterize shear stress. Turbulent kinetic energy is defined as half a trace of the correlation tensor $u'_iu'_j$:

$$k = \frac{1}{2} u'_i \cdot u' = \frac{1}{2} u'_iu'_i.$$  \hspace{1cm} (5.11)

We can divide tensor $u'_iu'_j$ into an isotropic part $i_j$ and a deviatory anisotropic part $a_{ij}$:

$$u'_iu'_j = i_j + a_{ij},$$  \hspace{1cm} (5.12)

where the isotropic part is

$$i_j = \frac{2}{3} k \delta_{ij},$$  \hspace{1cm} (5.13)

and the non-isotropic part is

$$a_{ij} = u'_iu'_j - \frac{2}{3} k \delta_{ij}.$$  \hspace{1cm} (5.14)

The isotropic part of Reynolds stress can be included into modified mean pressure – see (5.20).

We can mathematically describe the behavior of fluctuation quantities by applying to N-S equations Reynolds decomposition for velocity and pressure and subsequently subtracting the Reynolds equations for mean values. Then we reach:

$$\frac{\partial u'_i}{\partial t} + u_k \frac{\partial u'_i}{\partial x_k} + u'_k \frac{\partial u'_i}{\partial x_k} + \frac{\partial}{\partial x_i} (u'_iu'_j - u'_ku'_k) = -\frac{1}{\rho} \frac{\partial \rho'}{\partial x_i} + v \frac{\partial^2 u'_i}{\partial x_i^2}.$$  \hspace{1cm} (5.15)

Significance of individual terms:

$I$ time change of fluctuation velocity,

$II$ represents a mutual link between the field of mean velocity and the field of fluctuation,

$III$ is a non-linear term,

$IV$ is an effect of pressure fluctuations,

$V$ is a dissipation term.

In order to achieve an equation for a Reynolds stress tensor and for correlation tensor $u'_iu'_j$, we must multiply equation (5.15) by $u'_j$, add for changed indexes and perform the operation of averaging:

$$\frac{\partial u'_i}{\partial t} + u_k \frac{\partial u'_i}{\partial x_k} + u'_k \frac{\partial u'_i}{\partial x_k} = -\frac{1}{\rho} \frac{\partial \rho'}{\partial x_i} + v \frac{\partial^2 u'_i}{\partial x_i^2}.$$  \hspace{1cm} (5.16)

The physical significance of individual terms in this equation is as follows:

$I$ is the time change of the local correlation tensor,

$II$ represents advection of Reynolds stress through a mean flow (not total!),

$III$ this term represents interaction between the mean and fluctuation components of the flow and is related to the production of Reynolds stress,

$IV$ represents advection in relation to the fluctuation part of the flow,

$V$ represents the effect of pressure, and finally
VI represents diffusion and dissipation resulting from viscosity, which is expressed mainly in small scale perturbations.

The equation for Reynolds stress (5.16) theoretically could solve the problem of the inability to close Reynolds equations. However, in fact this is not the case, because this equation contains many of other unknown quantities – 2\textsuperscript{nd} order correlations with pressure (term \( V \)) and 3\textsuperscript{rd} order correlations (term \( IV \)).

Turbulent flows are characterized by a non-zero value of vorticity. In the case of flow without vorticity, the mean and fluctuation vorticity part is zero. When we multiply the vorticity fluctuation part by the velocity fluctuation part and perform averaging, we reach

\[
u_k' \left( \frac{\partial u_k'}{\partial x_i} - \frac{\partial u_k'}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left( \frac{1}{2} u_k' u_k' \right) - \frac{\partial}{\partial x_k} u_k' u_k' = 0 .\] (5.17)

For flow without vorticity we then reach the Corrsin-Kistler equation

\[rac{\partial}{\partial x_i} u_k' u_k' = \frac{\partial k}{\partial x_i} .\] (5.18)

We can see that in flow without vorticity, Reynolds stress \( u_k' u_j' \) plays the same role as isotropic stress \( k\delta_{ij} \), which can be included into modified pressure. It is apparent from this that in the case of flow without viscosity, Reynolds stress has no effect on the mean velocity field.

Let us also notice certain cases, which can be regarded in the sense of mean quantities as two-dimensional and planar. These are flows in relation to which we have geometrically planar boundary conditions, but it is necessary to realize that the instantaneous fields of quantities in turbulent flow are always three-dimensional.

### 4.1.2. Possibilities for solving Reynolds equations

During practical use of Reynolds equations, we must cope with the problem of the inability to close their system. We have available 3 component Reynolds equations and a continuum equation, meaning a total of 4 equations. Although 10 are unknown, and these are 3 components of mean velocity, mean pressure and 6 independent components of Reynolds tensor stress.

There have been attempts to solve this problem since Reynolds’ era, when some basic concepts were proposed. The first simple models do not sufficiently take into consideration the physical behavior of the system and do not generally result in very good results, but they are clear and transparent. During proper use, they can put out positive results. The concept of modern methods of solving Reynolds equations is based mainly on historical models, and therefore in this case we are presenting certain useful concepts.

One of the classic methods of modeling Reynolds stress is a turbulent viscosity hypothesis which was introduced even in 1877 by Boussinesq. This method uses an analogy of Newton law for expressing shear stress in fluid.

According to this hypothesis, stress in fluid, which corresponds to deviation anisotropic parts of Reynolds stress (5.14) is proportional to the mean velocity of deformation of fluid particles. The proportionality constant is “turbulent viscosity” or “eddy viscosity” \( \nu_r \):

\[-\rho u_i u_j' + \frac{2}{3} \rho k \delta_{ij} = \rho \nu_r \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\] (5.19)

Reynolds equations then transfer to the formulation

\[rac{Du_i}{Dt} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_i} \left[ \nu_{\text{eff}} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \right] \] (5.20)

where \( \nu_{\text{eff}} (x,t) = \nu + \nu_r (x,t) \) is the coefficient of effective viscosity. Let us notice that these equations formally have the same form as N-S equations in which time mean velocity figures. The coefficient of molecular viscosity is replaced by an effective coefficient, and the pressure is then modified by mean pressure \( \bar{P} \).
\[ \bar{P} = \bar{p} + (2/3) \rho k \] (5.21)

Determining turbulent viscosity, which is a function of position and time, remains a problem. Turbulent viscosity is connected with the typical velocity value \( U \) and with the size of the largest turbulent vortices \( L \) via relationship \( \nu_t \sim U \cdot L \). Generally, turbulent viscosity is a time function, and during practical application it is expected that changes in turbulent viscosity can be neglected in time. For distribution of the values of turbulent viscosity in space, recommendations exist that are valid for a certain category of problems.

In 1925, Prandtl introduced a special scale called a mixing length. With the use of a hypothesis formulated by Boussinesq and an analogy with molecular diffusion, which expresses velocity gradients for smaller scales than the mean free paths of molecules, Prandtl assumed that turbulent diffusion exists, which pushes away velocity fields for scales smaller than the mixing length \( l_{\text{mix}} \). Then the Reynolds stress tensor can be rewritten as a turbulent diffusion term. The relationship between the mixing length and turbulent viscosity in the boundary layer near the solid wall is

\[ \nu_t = l_{\text{mix}}^2 \frac{\partial \bar{u}_k}{\partial x_i}, \] (5.22)

where \( \bar{u}_k \) is the mean velocity along the wall, and \( x_i \) is the coordinate perpendicular to the wall.

Algebraic models of turbulence are based on the idea of a mixing length. However, if we consider flow during high values of Reynolds number, the hypothesis using the analogy of kinetic theories of gases ceases to function. Molecular diffusion can be modeled through the linear diffusion Laplace equation, because diffusion motions are separated from motions of large scales. However, this does not apply for fully developed turbulent flow, in which a non-linear advective term dominates over a diffusion term and contains fluctuation of all scales. Therefore, large and small scales of motion are not separated here. This situation is the main obstacle for attempts to modeling turbulence with the help of momentum equations, and the problem posed by the inability to close them remains unsolved.

We will present other possible ways of solving the problem of inability to close Reynolds equations in the chapter regarding modeling of turbulent flows with the help of Reynolds equations.

### 4.2. Energy balance

Energy in a unit volume of fluid is comprised of kinetic energy of flow, which is equal to \( \frac{1}{2} \rho u_i^2 \), and of internal energy \( e \rho \), which is related to sub-atomic forces and to relative motion of individual molecules given the macroscopic motion of fluid. On the outside we can measure internal energy with the help of the temperature and specific heat of the fluid. In the following paragraphs, we will focus only on the balances of kinetic energy in fluid.

Kinetic energy for mass unit \( k = \frac{1}{2} u_i u_k \) does not contain derivatives, because it can be assumed that it is determined mainly based on large scale structures. However, dissipation velocity

\[ \Delta = \frac{1}{2} u_i \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) \] (5.23)

depends on spatial derivatives of velocity, and the largest values will obviously be reached in smaller scales. Therefore, kinetic energy is basically connected to large scales, which dissipate as a result of viscosity in the area of small scales. In order for this to occur, there must be a connection between both values, which is mediated by a transfer of energy from large scales to small ones.

Let us now apply the Reynolds decomposition to a formula for calculating kinetic energy.
The mean kinetic energy will then be
\[ \overline{k} = \frac{1}{2} u_i u_k = \frac{1}{2} u_i u_k + \frac{1}{2} u_i' u_k'. \]  

(5.24)

The terms on the right side represent mean stream energy and mean energy of turbulence. Mean energy of turbulence is an important quantity, which quantifies the intensity of turbulent motions. A mean quadratic value is also introduced
\[ q^2 = u_i' u_k'. \]  

(5.25)

along with a mean quadratic deviation
\[ \sigma_q = \sqrt{q^2 / 3} = \sqrt{(u_i'^2 + u_j'^2 + u_k'^2) / 3}. \]  

(5.26)

Mean energy of turbulence characterizes the motions of large scales. The turbulent Reynolds number
\[ \text{Re}_T = \sigma_q L / \nu \]  

(5.27)

defined for the larger scales \( L \) then characterizes the significance of viscosity for the largest energy structures.

### 4.2.1. Mean stream energy

Let us rewrite the Reynolds equations (5.8) into the following form
\[ \frac{Du_i}{Dt} = \frac{\partial T_{ik}}{\partial x_k}, \]  

(5.28)

where
\[ T_{ij} = \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{p}{\rho} \delta_{ij} - u_i' u_j'. \]  

(5.29)

is the effective tensor of shear stress divided by density.

We will further predict flow in a certain space limited by solid walls, on which the flow velocity is zero. Let us multiply the equation (5.28) \( \overline{u}_k \) and integrate it through the volume of fluid \( V \), which moves at the mean stream velocity. Then we can substitute the substantial derivative with an ordinary one and factor it out
\[ \frac{d}{dt} \left[ \frac{1}{2} u_i u_i \right] dV = \int \overline{u}_i \frac{\partial T_{ij}}{\partial x_j} dV = \int \left( \frac{\partial}{\partial x_i} (\overline{u}_k T_{ik}) \right) dV - \int T_{ij} \frac{\partial \overline{u}_j}{\partial x_i} dV. \]  

(5.30)

The first term on the right side of the equation is 0, and therefore we can rewrite the entire equation as follows:
\[ \frac{d}{dt} \left[ \frac{1}{2} u_i u_i \right] dV = \int \left( \frac{\partial u_i' u_k'}{\partial x_i} \right) dV - \frac{1}{2} \int \left( \frac{\partial \overline{u}_i}{\partial x_i} + \frac{\partial \overline{u}_i}{\partial x_i} + \frac{\partial \overline{u}_i}{\partial x_i} + \frac{\partial \overline{u}_i}{\partial x_i} \right) dV. \]  

(5.31)

The physical significance of the individual terms is as follows:

- **I** represents the rate of change of the mean stream energy,
- **II** is based on the link of the mean stream and turbulent motions, and
- **III** represents viscous dissipation of the mean stream.

The link term between the mean stream and turbulence **II** represents energy collected from the mean stream for turbulence. This term is related to the Reynolds stress and is typically much greater than viscous dissipation of the mean flow.

### 4.2.2. Total energy

The process for deriving relationships for total energy is similar to the process for the balance of mean stream energy performed in the previous paragraph, but we work from N-S equations. We multiply them by \( u_i \) and integrate them through the flow area limited by solid walls.
\[
\frac{d}{dt} \int \frac{1}{2} u_k u_k dV = -\frac{1}{2} \nabla \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) dV = -\int \Delta dV ,
\]  

(5.32)

Where \( \Delta \) is the rate of dissipation of energy in relation to viscosity for a mass unit. The equation gives us the total (integrated) rate of a change of energy, and it includes in itself also the influence of viscous shear stress and the influence of pressure and convection. If we use the Reynolds decomposition for \( \Delta \), then for its mean value we reach

\[
\bar{\Delta} = \frac{1}{2} \nabla \left( \frac{\partial \bar{u}_k}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_k} \right) \left( \frac{\partial \bar{u}_k}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_k} \right) + \frac{1}{2} V \left( \frac{\partial u'_k}{\partial x_i} + \frac{\partial u'_i}{\partial x_k} \right) \left( \frac{\partial u'_k}{\partial x_i} + \frac{\partial u'_i}{\partial x_k} \right) .
\]  

(5.33)

We have distributed the total mean dissipation into mean and fluctuation components, therefore turbulent. The turbulent component tends to be referred to as dissipation rate \( \varepsilon \) and plays an important role in the theory of turbulence

\[
\varepsilon = \frac{1}{2} V \left( \frac{\partial u'_k}{\partial x_i} + \frac{\partial u'_i}{\partial x_k} \right) \left( \frac{\partial u'_k}{\partial x_i} + \frac{\partial u'_i}{\partial x_k} \right)
\]  

(5.34)

and its mean value

\[
\bar{\varepsilon} = \frac{1}{2} V \left( \frac{\partial \bar{u}'_k}{\partial x_i} + \frac{\partial \bar{u}'_i}{\partial x_k} \right) \left( \frac{\partial \bar{u}'_k}{\partial x_i} + \frac{\partial \bar{u}'_i}{\partial x_k} \right).
\]  

(5.35)

We will now decompose the terms in equation (5.32) to a part corresponding to the mean stream and the turbulent component and perform the operation of averaging in time. For total mean energy, the following applies:

\[
\frac{d}{dt} \int \frac{1}{2} u_k u_k dV = -\frac{1}{2} V \left( \frac{\partial \bar{u}_k}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_k} \right) \left( \frac{\partial \bar{u}_k}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_k} \right) \left( \frac{\partial \bar{u}'_k}{\partial x_i} + \frac{\partial \bar{u}'_i}{\partial x_k} \right) \left( \frac{\partial \bar{u}'_k}{\partial x_i} + \frac{\partial \bar{u}'_i}{\partial x_k} \right) dV .
\]  

(5.36)

This formula shows us how the total mean stream energy dissipates. The dissipation rate is comprised of a component characterizing viscous dissipation of the mean stream (the first term on the right side) and of a component characterizing the turbulent viscous dissipation (the second term on the right side).

In practical cases, when the Reynolds number is sufficiently high, the component characterizing the dissipation of the mean stream is negligible compared to the turbulent component, and therefore the following applies:

\[
\frac{1}{2} V \left( \frac{\partial \bar{u}'_k}{\partial x_i} + \frac{\partial \bar{u}'_i}{\partial x_k} \right) \left( \frac{\partial \bar{u}'_k}{\partial x_i} + \frac{\partial \bar{u}'_i}{\partial x_k} \right) \ll \frac{1}{2} V \left( \frac{\partial u'_k}{\partial x_i} + \frac{\partial u'_i}{\partial x_k} \right) \left( \frac{\partial u'_k}{\partial x_i} + \frac{\partial u'_i}{\partial x_k} \right).
\]  

(5.37)

This is because the fluctuation gradients of velocity \( \partial u'_j/\partial x_j \) acquire high and increasingly higher values with the growing Reynolds number, while gradients of mean velocity do not. These gradients acquire the highest values in small scales of turbulent motions. One of the exceptions is the area of the boundary layer in direct proximity to the wall, where a viscous sublayer is formed which has a laminar character even in the case of a turbulent boundary layer. Gradients of mean velocity in a direction perpendicular to the wall dominate, while gradients connected to fluctuations are practically missing here and the relationship (5.37) does not apply here. However, this case relates to very sparsely limited areas in direct proximity to the wall, and everywhere else relationship (5.37) applies.

Mutual energy connection of the mean stream and turbulence is expressed in the first term on the right side of equation (5.31), which represents the production of turbulence, but it does not appear in the total energy balance.

### 4.2.3. Energy of turbulence

We can obtain the equation for turbulent kinetic energy from equations (5.16) by positioning \( j = i \) and dividing by 2, and we can express turbulent kinetic energy for the mass unit with the help of (5.24) and (5.25):
\[
\frac{\partial (q'^2/2)}{\partial t} + u_k \frac{\partial (q'^2/2)}{\partial x_k} = -u_k' \frac{\partial u_i}{\partial x_k} - \frac{\partial (q'^2 u_i'/2)}{\partial x_k} - \frac{1}{\rho} u_i' \frac{\partial p'}{\partial x_i} + \nu u_i' \frac{\partial^2 u_i'}{\partial x_j \partial x_j}. \tag{5.38}
\]

This very important equation in the theory of turbulence contains several terms:

- \(I\) represents the rate of the change of turbulent energy in time in the particular point (for steady, well developed flows it is 0),
- \(II\) is the convection of turbulent energy through the mean stream. The sum of terms \(I\) and \(II\) gives the rate of the time change of turbulent energy for the mass unit in the point by moving at a mean velocity,
- \(III\) is the production of turbulent energy created by mutual effects between the mean stream and turbulence, and this term is most often positive (turbulent energy is supplied from the mean stream), but in exceptional cases it can be negative (turbulent energy is consumed),
- \(IV\) is advective transport of turbulent energy through fluctuation motions,
- \(V\) is the motion of turbulent energy as a result of pressure (work performed through pressure fluctuations),
- \(VI\) represents viscous diffusion and dissipation.

We can rewrite term \(VI\) as follows:

\[
u u_i' \frac{\partial^2 u_i'}{\partial x_j \partial x_j} = \nu \frac{\partial}{\partial x_k} \left[ u_i' \left( \frac{\partial u_i'}{\partial x_k} + \frac{\partial u_i'}{\partial x_i} \right) \right] - \bar{\varepsilon}, \tag{5.39}
\]

The first term on the right side of the equation (5.39) represents viscous transport, and the second represents the dissipation of turbulent energy. The integral of the turbulent transport term across the entire flow area is obviously zero (zero divergence) and therefore does not contribute to a change of total energy. This term is often described as a diffusion term, because for homogeneous turbulence it acquires zero value, and in non-homogeneous turbulence it represents the “diffusion” of kinetic energy. Turbulent diffusion is necessary in the equation for kinetic turbulent energy to model, because it is represented by a triple correlation of velocity fluctuations. In practice, this term is often much smaller than a dissipation term, and therefore it can be ignored (an exception is the viscous layer mentioned above). The dissipation term plays a very important role in each turbulent flow.

We can adjust equation (5.38) with the help of relationship (5.39), and assuming that the fluid is incompressible and the continuity equation is valid for mean velocity and for fluctuations, we arrive at the following practical useful formulation of the energy equation:

\[
\frac{\partial (q'^2/2)}{\partial t} = -u_k' \frac{\partial u_i}{\partial x_k} - \frac{\partial (q'^2 u_i'/2)}{\partial x_k} - \frac{1}{\rho} u_i' \frac{\partial p'}{\partial x_i} - \nu u_i' \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial u_i'}{\partial x_i} \right). \tag{5.40}
\]

In equation (5.40) the following applies:

- \(I\) is production of turbulent energy,
- \(II\) is dissipation of turbulence,
- \(III\) is a representation of the effect of convection via a mean stream,
- \(IV\) is a relationship with diffusion, and pressure fluctuations contribute to increasing the isotropy and homogeneity of turbulence,
- \(V\) is transport of turbulent energy – sum of terms \(III\) and \(IV\).

Through integration of transport term \(V\) over the whole flow area to the walls, we arrive at 0 (the theorem of zero divergence); the process therefore involves redistribution of energy from one place to another, and the total energy in relation to this term does not change. The total energy of the flow given by the integral through the flow area is therefore determined by the difference of the integrals of the production and dissipation term. For statistical stationary and developed turbulent flows, the production and dissipation of turbulent energy in an equilibrium must be obvious, and therefore the right side of equation (5.40) will be zero.
Through a simple mathematical adjustment of transport term $V$, when we multiply it by the density and perform integration through a random fixed volume, we can interpret the result as a vector determining the mean flow of turbulent energy in relation to convections through mean stream and diffusion of turbulent energy

$$\frac{1}{2} \rho q^2 u_i + u_i (\frac{1}{2} \rho q^2 + p') - \mu u_i \left( \frac{\partial u'_i}{\partial x_k} + \frac{\partial u'_k}{\partial x_i} \right).$$  \hspace{1cm} (5.41)

This vector characterizes the non-homogeneity of the turbulent flow.

For a homogeneous turbulent flow we can arrive at a simplified equation of energy without the transport term

$$\frac{\partial \left( \frac{q^2}{2} \right)}{\partial t} = -u_i u'_i \frac{\partial u_k}{\partial x_i} - \varepsilon.$$ \hspace{1cm} (5.42)

The production term causes the occurrence and/or loss of turbulent energy depending on its sign. Without gradients of the mean velocity field, production and turbulent energy would continue to fall. Dissipation term $\varepsilon$ is always positive and always causes the outflow of energy via a known mechanism through turbulent motions of small scales, which generate the greatest values of spatial gradients and therefore also the most intensive dissipation.

For isotropic turbulence, it applies that none of the directions is special or given priority in any manner. For the Reynolds stress tensor $u_i u'_j$, they must have the same components, so that we can perform random rotation of the coordinate system. It is apparent from the tensor calculus that the particular tensor must have a formulation that can be expressed as the product of the scalar constant and a Kronecker tensor, and the scalar constant in our case is a quadrate of the standard deviation in the velocity vector modulus $\sigma_u^2 = u_i u'_i / 3$. For isotropic turbulence, therefore $u_i u'_j = \sigma_u^2 \delta_{ij}$ applies, and the Reynolds stress tensor is diagonal with all identical diagonal elements. This property can be used when verifying the hypothesis about the isotropy of the specific turbulent flow.

### 4.2.4. Energy dissipation rate

We can anticipate turbulent flows if there is a sufficiently high Reynolds number, meaning outside of the viscous sublayer. As has already been stated above, the rate of dissipation $\Delta$ (see equation (5.23)) is a function of spatial derivative of velocity, and for its size the smallest turbulent structures are decisive, and we can ignore the effect of the gradients of mean flow here and take into consideration only the turbulent part of dissipation rate $\varepsilon$. We can adjust the equation (5.35) with the help of the continuity equation for velocity fluctuation, which yields the following identity:

$$\frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_k} = \frac{\partial^2 u_i u_i}{\partial x_i \partial x_i},$$  \hspace{1cm} (5.43)

Into the following formulation

$$\varepsilon = \nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_k} + \nu \frac{\partial^2 u_i u_i}{\partial x_k \partial x_i}.$$ \hspace{1cm} (5.44)

Through simple arithmetic operations, we can also arrive at the relationship for a mean turbulent dissipation rate expressed with the help of vorticity fluctuations.

$$\varepsilon = \nu \omega'_i \omega'_i + 2\nu \frac{\partial^2 u_i u_i}{\partial x_k \partial x_i}.$$ \hspace{1cm} (5.45)

For homogeneous turbulence, the second term falls away on the right side of equations (5.44) and (5.45), and for the dissipation rate we arrive at the simplified form

$$\varepsilon = \nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_k} = \nu \omega'_i \omega'_i.$$ \hspace{1cm} (5.46)
It has been shown that a simplified form for dissipation rate (5.46), which is exactly valid only for homogeneous turbulence, applies with sufficient accuracy in most turbulent flows, assuming the Reynolds number is sufficiently high. This is because the spatial derivatives of instantaneous velocity contained in the first term basically are larger than the derivatives of mean values, which is the case for Reynolds stress \( \overline{u'u'} \) in the second term. Therefore, we can ignore the second term.

For isotropic turbulence, a simplified formula applies for calculating the mean turbulent rate of dissipation:

\[
\overline{\varepsilon} = 15\nu \left( \frac{\partial u'}{\partial x_i} \right)^2,
\]

(5.47)

where direction 1 is a random direction (for the isotropic flow field all directions are equivalent) and this index is not an summation index.

Note: In the next chapters, especially in Chapter 7, we will label the turbulent component of dissipation rate \( \varepsilon \) without a sign of averaging, but we will always understand the mean value in time, not its instantaneous value.

### 4.2.5. Mean vorticity

Vorticity vector \( \boldsymbol{\omega} \) was already previously defined as the rotation of velocity vector \( \boldsymbol{u} \). For the vorticity field, it applies that its divergence is an identical zero value (see (12.10)):

\[
\nabla \cdot \boldsymbol{\omega} = 0.
\]

(5.48)

We can use the Reynolds decomposition for velocity (5.1) also for vorticity

\[
\boldsymbol{\omega} = \overline{\boldsymbol{\omega}} + \omega'.
\]

(5.49)

Since equation (5.48) is linear, an analogical relationship must apply also for both components separately, and so the divergence of the field of mean vorticity and its fluctuations has a zero value; in other words all of these vector fields are solenoidal.

We already identified the Helmholtz equation for instantaneous vorticity previously, and now we are rewriting it into the component formulation

\[
\frac{\partial \omega_i}{\partial t} + u_k \frac{\partial \omega_i}{\partial x_k} = \omega_k \frac{\partial u_j}{\partial x_k} + \nu \frac{\partial^2 \omega_i}{\partial x_i \partial x_k}.
\]

(5.50)

The physical interpretation of individual terms is from left to right: non-stationary term, convective term, term characterizing stretching of vortices and viscous diffusion.

Now let us carry out the Reynolds decomposition of vorticity and apply the averaging operation to the Helmholtz equation. By doing so, we reach the equation for mean vorticity

\[
\frac{\partial \overline{\omega_i}}{\partial t} + u_k \frac{\partial \overline{\omega_i}}{\partial x_k} = \overline{\omega_k} \frac{\partial u_j}{\partial x_k} + \nu \frac{\partial^2 \overline{\omega_i}}{\partial x_i \partial x_k} + \left[ \omega_k \overline{u'_j} - \omega'_k \overline{u_j} \right] + \nu \frac{\partial^2 \omega_i}{\partial x_i \partial x_k}.
\]

(5.51)

- **I** obviously characterizes a non-stationary condition,
- **II** is advection via a mean stream,
- **III** stretching related to mean stream,
- **IV** turbulent advection,
- **V** turbulent stretching and
- **VI** viscous diffusion.

Term **IV** and **V** represent the effect of turbulence on the field of mean vorticity, and the other terms are not affected by turbulence.

The fluctuation component of vorticity is described with the following equation

\[
\frac{\partial \omega_i'}{\partial t} + u_k \frac{\partial \omega_i'}{\partial x_k} = \omega_k' \frac{\partial u_j}{\partial x_k} - \omega_k' \frac{\partial u_j}{\partial x_k} + \nu \frac{\partial^2 \omega_i'}{\partial x_i \partial x_k}.
\]

(5.52)
We can gain a better physical impression of the intensity of vorticity by introducing a new physical quantity, which is usually referred to as “enstrophy”. Enstrophy is an important quantity characterizing the level of non-stationary swirling of the flow. Unlike vorticity, it involves a scalar. Enstrophy $\zeta$ is defined as variance of vorticity

$$\zeta = \frac{1}{2} \omega^2 = \frac{1}{2} \omega_k^2. \tag{5.53}$$

The relationship of enstrophy $\zeta$ to vorticity $\omega$ is equivalent to the relationship of kinetic energy $k$ and the velocity vector $u$ – compare with equation (5.11). Total enstrophy is then analogically expressed with the help of total vorticity.

The following identity also applies:

$$\frac{\varepsilon}{\nu} = \frac{\zeta}{\nu} + \frac{\partial u'_k}{\partial x_i} \frac{\partial u'_k}{\partial x_i}, \tag{5.54}$$

which indicates the relationship between the dissipation rate and enstrophy. In homogeneous turbulence, it applies that the second term on the right side of expression (5.54) is equal to 0. It has been shown that even in shear areas, which obviously are not homogeneous, formula $\zeta = \varepsilon/\nu$ can be considered with very good accuracy. This situation offers the option of use of local values of enstrophy in turbulence models instead of dissipation rate.

We can break total enstrophy down into 2 parts with the help of Reynolds vorticity decomposition.

$$\frac{1}{2} \omega_k^2 = \frac{1}{2} \omega_k^2 + \frac{1}{2} \omega_k^2. \tag{5.55}$$

Here we have arrived at an equation for enstrophy, which we get by multiplying the equation for vorticity and multiplying the vorticity vector. For mean enstrophy, we start from equation (5.51)

$$\frac{\partial (1/2) \overline{\omega_k \omega_k}}{\partial t} + u_i \frac{\partial (1/2) \overline{\omega_k \omega_k}}{\partial x_i} =$$

$$= \overline{\omega_k \omega_k} \frac{\partial u'_k}{\partial x_i} - \overline{\omega_k} \frac{\partial \overline{\omega_k}}{\partial x_i} \left[ \frac{\partial u'_k}{\partial x_i} \right] - \nu \frac{\partial^2 (1/2) \overline{\omega_k \omega_k}}{\partial x_i^2} - \nu \frac{\partial \overline{\omega_k \omega_k}}{\partial x_i} \frac{\partial \overline{\omega_k}}{\partial x_i} \frac{\partial \overline{\omega_k}}{\partial x_i} \tag{5.56}$$

$I$ characterizes a non-stationary condition,

$II$ is advection via a mean stream,

$III$ stretching related to mean stream,

$IV$ is turbulent advection and

$V$ turbulent stretching of vortices,

$VI$ is an term of viscous transport, and

$VII$ is then viscous dissipation.

When coming up with the equation for fluctuation components of enstrophy, we start from equation (5.52)

$$\frac{\partial (1/2) \overline{\omega'_k \omega'_k}}{\partial t} + u_i \frac{\partial (1/2) \overline{\omega'_k \omega'_k}}{\partial x_i} =$$

$$= \overline{\omega'_k \omega'_k} \frac{\partial u'_k}{\partial x_i} + \overline{\omega'_k \omega'_k} \frac{\partial u'_k}{\partial x_i} - \nu \frac{\partial^2 (1/2) \overline{\omega'_k \omega'_k}}{\partial x_i^2} - \nu \frac{\partial \overline{\omega'_k \omega'_k}}{\partial x_i} \frac{\partial \overline{\omega'_k \omega'_k}}{\partial x_i} \tag{5.57}$$

The terms represent:

$I$ again characterizes non-stationary condition,

$II$ is advection via a mean stream,

$III$ then represents a link between the mean flow and turbulence,
IV is turbulent stretching related to velocity fluctuations,
V is transport related to non-homogeneousness and
VI is viscous dissipation.

Individual terms in this equation have very variable sizes depending on the Reynolds number. If we expect very high Reynolds number values, then it will be shown that on the right side of equation (5.57) terms IV and VI are much larger than the others. For fluctuation of the enstrophy component, we arrive at a simplified equation, which corresponds to a situation without the presence of gradients in the mean stream:

\[
\frac{\partial (1/2) \omega^2}{\partial t} + u_j \frac{\partial (1/2) \omega^2}{\partial x_j} = \omega_j \omega_k \frac{\partial u_k}{\partial x_j} - \nu \frac{\partial \omega_k}{\partial x_j} \frac{\partial \omega_l}{\partial x_l}. \tag{5.58}
\]

This equation describes the dynamics of small scales, but it says nothing about behavior in the area of large scales. For homogeneous turbulence, the second term on the left side of equation (5.58) also has a zero value. We can go even further; if the turbulent flow is entirely developed during a very high Reynolds number value, then we can even ignore the entire left side of equation (5.58), and we will arrive at an equilibrium between the turbulent stretching of vortices and their disintegration as a result of viscosity. For achievement of an equilibrium, the term representing stretching must be positive, and therefore stretching of vortices must actually occur, not their shrinkage.

### 4.3. Main problem of turbulence

It is apparent from Reynolds equation equation reference goes here(5.8) that in a turbulent flow, a change of mean momentum depends not only on the forces of molecular viscosity as with laminar flows, but also on the transfer of momentum through fluctuation motions. This creates Reynolds stress, which has decisive significance for the behavior of a fluid in a turbulent state. Its creation is related to the transfer of momentum with the help of turbulent vortices contained in the flow fluid. Each of the particles from which these vortices are comprised contains a large number of molecules and moves to a distance that greatly exceeds the free path of the molecules. Therefore, the transfer of both the scalar and vector properties of a fluid using turbulent mechanisms is much more effective than a transfer with the help of random motions of molecules, which determine viscous forces. Turbulent shear stress is therefore much greater in turbulent flow than viscous stress. This applies in shear areas everywhere except for in thin layers near the surface – viscous sublayers.

Unfortunately the system of Reynolds equations is not closed – the number of equations is smaller than the number of variables contained in them. We have available 3 Reynolds equations plus a continuity equation. From unknowns we have 3 components of mean velocity, mean pressure and 6 components of Reynolds stress. So there are 4 equations for 10 unknowns. The set needs to be closed with the help of other equations. However, if we write equations for Reynolds stress (5.16), the situation is even worse, because in doing so we get a new set of unknown moments of the 2nd and 3rd orders. Then although we have 10 equations, we also have 75 unknowns.

The problem with the inability to close a set of Reynolds equations is often called the “main problem of turbulence”. This problem can be solved with the help of various simplifications and predictions about the properties of Reynolds stress. These collectively involve phenomenological models accepted ad hoc without a deeper understanding of the physical basis of the phenomena that cause this behavior. These models are created with the help of a set of experimental data, and the model is not much more than a mere regression of the data. With the adoption of such a model, fundamental information contained in N-S equations is often sacrificed. These models are labeled as artifacts at various levels of the mathematical apparatus. The result is that they have very limited validity and are usable only for the situations from which they were derived and/or for very similar situations with only small modifications. Generally, each category of cases of turbulent flow of fluid requires its own such model or variant.
4.4. Bernoulli equation

The use of a Bernoulli equation is very common in technical applications. Now we will derive this equation from general N-S equations while adopting certain presumptions and simplifications, and in doing so we will clarify the conditions for the validity of the Bernoulli equation and its usability in turbulence.

We should assume an inviscid, incompressible fluid during stationary flow. The behavior of such a fluid can be described using an Euler equation for stationary flow:

\[(u \cdot \nabla)u = -\frac{\nabla p}{\rho} + g,\]  \hspace{1cm} (5.59)

where \( g \) is a vector of external acceleration. The gravitational acceleration of \( g \) moves against the direction of axis \( x_3 \), and therefore we can express this term as follows:

\[g = -g e_3,\]  \hspace{1cm} (5.60)

where \( e_3 \) is the unit vector in the direction of axis \( x_3 \). Additionally for vector \( u \) the following vector identity applies (see also (12.11))

\[\nabla \left( u \cdot \nabla \right)u = \frac{1}{2} \nabla \left( u \cdot u \right) - u \times (\nabla \times u).\]  \hspace{1cm} (5.61)

We can then rewrite equation (5.59) as follows:

\[\frac{1}{2} \nabla \left( u \cdot u \right) - u \times (\nabla \times u) = -\frac{\nabla p}{\rho} - g e_3.\]  \hspace{1cm} (5.62)

After further adjustments, we get

\[\frac{\nabla p}{\rho} + \frac{1}{2} \nabla \left( u^2 \right) + g e_3 = u \times (\nabla \times u),\]  \hspace{1cm} (5.63)

where \( u \) is the modulus of velocity vector \( |u| \).

Let us now observe the development of this equation during displacement along the streamline. We will therefore multiply the equation by the elementary vector \( ds \), which lies on the streamline

\[\frac{\nabla p}{\rho} \cdot ds + \frac{1}{2} \nabla \left( u^2 \right) \cdot ds + g e_3 \cdot ds = [u \times (\nabla \times u)] \cdot ds,\]  \hspace{1cm} (5.64)

\( ds \) has the direction of the streamline, which means that it is also parallel with the local velocity vector \( u \). Vector \( u \times (\nabla \times u) \) is then perpendicular towards this direction, and therefore the right side of equation (5.64) disappears. The following formula also applies for pressure:

\[\nabla p \cdot ds = \frac{\partial p}{\partial x_1} dx_1 + \frac{\partial p}{\partial x_2} dx_2 + \frac{\partial p}{\partial x_3} dx_3 = dp.\]  \hspace{1cm} (5.65)

The same formula applies for the velocity quadrate. Therefore, we can further rewrite equation (5.64) as

\[\frac{dp}{\rho} + \frac{d \left( u^2 \right)}{2} + g dx_3 = 0.\]  \hspace{1cm} (5.66)

We can now integrate this equation along the streamline, and for an incompressible fluid we arrive at the known relationship also called the Bernoulli equation or the Bernoulli theory

\[\frac{p}{\rho} + \frac{u^2}{2} + g x_3 = H = \text{konst},\]  \hspace{1cm} (5.67)

where \( H \) is the Bernoulli constant.

The Bernoulli theory applies along streamlines (these are for stationary flow identical to trajectories) and along vortex lines (vector lines of field \( \omega \)). The Bernoulli constant \( H \) can be different for each line. Using substitution we reach the relationship

\[\nabla H = u \times \omega,\]  \hspace{1cm} (5.68)

which tells us that the quantity \( H \) is constant in the entire area if vector fields \( H \) and \( u \) are parallel; in other words the streamlines and vortex lines merge. Another possibility is a situation
when the flow lacks vorticity, meaning that vorticity equals zero everywhere and at every moment.

It is apparent from the above that the use of the Bernoulli equation is generally limited to laminar flows and does not apply for turbulent flows. However, for a homogeneous turbulent flow, this approach can be used for mean values of pressure and velocity, but it is necessary to perform correction with consideration for fluctuation. For the Bernoulli equation, we will use the mean velocity $U$ and modified pressure $\bar{P}$ according to (5.21).

### 4.5. Transport of passive scalar

A passive scalar is a quantity related directly to a fluid and which does not influence its properties that are important from the point of view of the dynamics of the fluid – meaning density and viscosity. A passive scalar can include temperature or the concentration of a certain admixture, assuming that we ignore the effect of changes of this quantity on the material quantities specified above.

The following equation applies for molecular diffusion of passive scalar $\theta$:

$$\frac{D\theta}{Dt} = \kappa \nabla^2 \theta,$$

in the event of diffusion of heat, when $\theta$ is the temperature of the fluid, we have on the left side substantial derivative $\theta$ with respect to time, and $\kappa$ is generally a molecular diffusivity, in this case it amounts to a thermal diffusivity (i.e. molecular thermal conductivity of fluid). This equation does not consider the dynamics of flow, only molecular effects – such as transport of the scalar across the flow field in the case of laminar flow.

Let us perform a qualitative analysis of this equation. Since $L$ is a typical dimension of area, $\Delta \theta$ is a temperature difference, and $T_m$ is the time scale of the molecular transfer of the scalar. Then the following formulation applies:

$$\frac{\Delta \theta}{T_m} \sim \kappa \frac{\Delta \theta}{L^2} \Rightarrow T_m \sim \frac{L^2}{\kappa}.$$  

The process of molecular diffusion is characterized by the Schmidt number

$$\text{Sh} = \frac{\nu}{\kappa},$$

And for temperature as a passive scalar, the Schmidt number is transformed into a Prandtl number. The typical value of a Prandtl number for regular fluids is in a range of units (air 0.7, water 7).

Let us now consider the second case of scalar transfer with the help of turbulent flow of fluid, which can be generated for example through pressure force during its local heating. The largest structures occurring in the area will have the regular area dimension $L$ and the typical velocity of the flow characterizing motions in fluid in these scales is $u$. The time scale characterizing the transfer of a scalar through turbulent motions of the largest scales will be

$$T_i \sim \frac{L}{u}.$$  

Let us now express the regular size of the ratio of time scales during molecular and turbulent transfer of the scalar:

$$\frac{T_m}{T_i} \sim \frac{L^2 u}{\kappa L} = \frac{Lu}{\kappa}. $$

If we accept the expectation that the Prandtl number is of order of units, then we can claim that the ratio of time scales during molecular and turbulent transfer of the scalar is of the same order as the Reynolds number, which characterizes the turbulent motion of fluid $Re=Lu/\nu$. Turbulent flow is characterized by high Reynolds number values (regularly $10^4$ and greater), and its value determines the enhancement of the process of diffusions in connection with turbulent motions.
5. Occurrence of turbulence

The occurrence of a turbulent flow in a certain area is conditioned on the creation of a certain situation in the examined area.

The process of the occurrence of turbulence causes break of the symmetries of N-S equations. Various more or less regular periodic structures are created, which cause the symmetry of shifts in time or in space to apply only for certain discreet values of such shifts, equaling the multiples of structure periods. A sufficiently large Reynolds number results in chaotic flow, and symmetries regain their validity, even if only in a statistical sense.

We can observe the transition of laminar flow into turbulence in shear areas of various types during an increase in the velocity of flow. The entire process is started with the occurrence of instability of the system due to a certain type of perturbation. Very small perturbations of all types are always present in the flow fluid, and this is due to the nature of the fluid, which is comprised of a very large number of particles – molecules. The motion, which is related to the molecular structure of fluids (Brownian motion) contains all modes and frequencies that the structure enables. It has the character if a white noise.

The loss of stability of laminar flow has as a result the occurrence of certain regular vortex sections and their growth. The first occurs according to a linear scenario, and non-linearities gradually begin to have an effect. This leads to mutual interactions of individual structures and their parts. During a certain phase, behavior occurs, which is typical for non-linear systems – the occurrence of deterministic chaos. Following homogenization in space, the process of the transfer to turbulence is ended.

We will now examine in detail the individual phases of this process. However, first we will present some historical information.

5.1. Reynolds’ experiment

Osborn Reynolds describes in his famous article from 1883 his experiment with examination of the stability of flow in a pipe with circular cross section during the outflow of fluid from a tank. A diagram of this experiment is shown in Figure 6.1.

Reynolds arrived at the following conclusions:
- If the flow velocity is sufficiently low, the color streak creates an almost perfect straight line inside the pipe (see Figure 6.1 (a)).
- If the water in the tank is not calm enough, the streak at a constant low velocity of flow can move in the pipe, but regular perturbations will not be created.
- If the flow velocity is gradually increased in small steps, then in a certain location in the pipe, but always in a sufficient distance from the inflow, random mixing of the coloring will

![Figure 6.1 – Reynolds experiment (original drawing)](image_url)
occur and the fluid will be equally colored in the entire cross section of the pipe (see Figure 6.1 (b)). Further increasing of the velocity will lead to approaching of the breaking point of the flow close to the inflow, it will never be reached. When using a spark discharge for illumination of the area of the broken flow, a system of more or less obvious vortices of various sizes is apparent (see Figure 6.1 (c)).

Reynolds correctly recognized the key task of the non-dimensional velocity parameter, which we now call the Reynolds number:

$$\text{Re} = \frac{Ud}{v},$$  \hspace{1cm} (6.1)

where $U$ indicates the magnitude of the velocity of flow in the pipe, $d$ is the diameter of the pipe and $v$ is kinematic viscosity. Reynolds looked for a “critical” value of this parameter, which separates regimes with and without the occurrence of breakdown in flow. During this search, he discovered that the entire matter is much more complicated. In his words: “… critical velocity is very sensitive to the presence of perturbations at the entrance to the pipe ... the hypothesis is therefore offered that the entire problem is a problem of instability of perturbations of a certain size and stability of smaller perturbations...”. Reynolds in the end through careful conducting of his experiment managed to gain stable flow for Reynolds numbers around 13,000. Later experiments were published in which stable flow through pipes during Reynolds numbers greater than 90,000 were observed. For comparison: under “regular” laboratory conditions without special measures, this value ranges around 2,000. Reynolds also discovered that the Reynolds number 2,000 is the limit when the originally turbulent flow returns to a laminar state. The described results of the experiments and the theoretical breakdowns of the problem show that developed flow through the pipe is stable due to the infinitesimal small perturbations for the random Reynolds number value. An analogy to flow in the pipe can be schematically illustrated in Figure 6.2 (a) represents a stable condition, (b) an indifferent condition at the edge of stability, (c) an unstable condition when a slight deviation would lead to instability, and finally (d) is a condition which is stable in view of the small perturbations and unstable in view of large ones.

5.2. Theory of stability

Let us shift now to examining stability issues using mathematical methods. We expect that perturbations in flow are infinitesimally small, so much so that any multiples of values by the size of the order of the perturbation can be ignored in our mathematical model. Therefore, it involves a linearized model, which in some cases very well illustrates the behavior of certain actual non-linear fluid systems near the limit of stability. The stability of the case, which is described above, Poiseuille flow through a pipe, cannot be investigated in this manner. However, it is possible to model well certain other cases, such as Kelvin-Helmholtz instability, Taylor instability, Bénard instability and certain other cases. However, it is necessary to emphasize that the linearized model can be used successfully only if the presumption is fulfilled, which is that flow perturbations are infinitesimally small. This practically means that with the help of a linearized model, we can successfully predict the boundary of stability of the system, but its behavior
beyond that boundary cannot be predicted, because to do so would require analysis of the complete non-linear model.

A basic approach during the examination of hydrodynamic stability is the use of the linear mathematical model. It is expected that laminar flow of fluid is subjected to the influence of certain perturbations, which can either be carried by flow (velocity fluctuation) or come from the wall (roughness). The linear theory examines the rate of changes of these perturbations in a laminar flow. A key question is whether the perturbations in time cease to exist or whether they grow over time. If they have a tendency to cease to exist, the flow is stable, but if they grow, then flow is unstable, and a basic expectation is created for its transition to a turbulent condition. The property of stability of flow in view of perturbations is due to flow conditions, and the Reynolds number especially has an effect. It has been shown that usually with a growing Reynolds number flow has a greater inclination towards instability.

Growth of perturbations in time and in space is conditioned on the creation of areas with absolute instability. An area that is absolutely unstable is demonstrated by a property according to which a locally inserted perturbation expands both in time and space. The result is that over a certain time frame the perturbation expands to the entire absolutely unstable area. An example of absolutely unstable areas is a wave behind a bluff body.

We can also recognize convective instability, which is characterized by convections of a perturbation with a flow in such a manner that after a certain amount of time the perturbation can no longer influence flow in its original location. A typical example of this is the limit layer occurring during parallel surrounding of a solid surface.

![Figure 6.3 - Convective and absolute instability](image)

Growth of perturbations during absolute and convective instability of a system is schematically displayed in Figure 6.3. On the vertical axis there is the time, and on the horizontal one there is the position of the perturbation in the flow field. Situation (a) corresponds to a stable condition – the perturbation in time shrinks. Figure 6.3(b) represents convective instability, (c) is instability at the boundary between the convective and absolute type, and (d) is absolute instability.

Dynamical systems in relation to stability are subject to hysteresis behavior. This means that the properties of the system are based not only on its condition, but also on the way it acquired a particular state. This means practically that if we examine the stability of the particular system depending on one selected parameter, typically a Reynolds number is involved, then for the particular value of this parameter the stability of the system depends on whether we reached the flow state by increasing or reducing the parameter or via another method (how fast,...).

### 5.2.1 Stability of inviscid flows

Let us consider the planar flow of inviscid fluid between two parallel walls according to Figure 6.4. The Figure shows a velocity profile, and in the case of inviscid flow the condition of zero velocity on the wall does not apply.
The flow of the fluid in this case is described by Euler equations, which in this case will be
\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p, \quad \nabla \cdot u = 0.
\]  
(6.2)

Their stationary solution is obviously \( u_0 = [U(x_2), 0, 0] \) parallel shear flow in the direction of axis \( x_1 \) with random course \( U(x_2) \) during constant pressure \( p_0 \) in the entire area. Let us now consider planar perturbations of velocity field \( u' = [u'_1, u'_2, 0] \) and similarly also \( p' \) perturbations of the pressure field. Here we see the relationships for the instantaneous values of velocity and pressure into (6.2) and the continuity equation, and following linearization and adjustments we reach the equation for fluctuation
\[
\frac{\partial u'_1}{\partial t} + U \frac{\partial u'_1}{\partial x_1} + u'_2 \frac{dU}{dx_2} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_1},
\]
\[
\frac{\partial u'_2}{\partial t} + U \frac{\partial u'_2}{\partial x_1} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_2},
\]
\[
\frac{\partial u'_1}{\partial x_1} + \frac{\partial u'_2}{\partial x_2} = 0.
\]  
(6.3)

We can use equation (6.3) for calculation of the size of perturbations in time when entering certain initial conditions. Such calculation would be relatively demanding and its results not very clear with limited ability to provide details about the basic properties of the system. Therefore, we will perform an analysis of stability with the help of a modal analysis of the system of equations (6.3).

Since the coefficients of equations (6.3) depend only on \( x_2 \), we can expect their solution, which is an exponential function of coordinate \( x_1 \) and time \( t \). Generally we should consider the perturbations in all 3 dimensions, but a detailed analysis of the problem can show that the planar perturbation is the least stable, and its formulation depends on coordinate \( x_3 \). These are the contents of the Squire theory. Therefore, we should consider the perturbations in the following formulation:
\[
u'_1(x_1, x_2, t) = \bar{u}_1(x_2) \exp[i(\kappa x_1 - \omega t)],
\]
\[
u'_2(x_1, x_2, t) = \bar{u}_2(x_2) \exp[i(\kappa x_1 - \omega t)],
\]
\[p'(x_1, x_2, t) = \bar{p}(x_2) \exp[i(\kappa x_1 - \omega t)].
\]  
(6.4)
A hat is used to label the amplitudes, \( i \) is an imaginary unit here, and \( \kappa \) and \( \omega \) are frequencies in the space and time domains. Of course only the real part of expressions (6.4) has a physical meaning. From the condition that the solution must be final for \( x_i \to \pm \infty \) it is apparent that constant \( \kappa \) is a real number. The frequency of fluctuations in time \( \omega \) is of course the generally complex number \( \omega = \omega_r + i \omega_i \). Expression (6.4) therefore represents the waves that move through the space in the direction of axis \( x_i \) at a phase velocity equal to \( \omega_r \) and which grow in time or shrink like expression \( \exp(\omega t) \). We can decide about the stability of these waves based on a criterion that monitors the value of an imaginary part of frequency \( \omega \). If \( \omega_i < 0 \), then the waves in time are stable and gradually cease to exist, for \( \omega_i > 0 \) waves grow and are unstable, \( \omega_i = 0 \) neutrally characterizes a stable condition at the edge of stability.

Now we will insert expressions (6.4) into equations (6.3) and we will achieve a set of three ordinary differential equations.

\[
\begin{align*}
-i(\omega - U \kappa) \dot{u}_i + U \ddot{u}_2 &= -\frac{1}{\rho} \frac{1}{\kappa} \hat{p}, \\
-i(\omega - U \kappa) \dot{u}_2 &= -\frac{1}{\rho} \hat{p}', \\
i \kappa \dot{u}_i + \ddot{u}_i &= 0.
\end{align*}
\]  
(6.5)

Here \( \cdot \) indicates derivative with respect to \( x_i \).

From equations (6.5) we can exclude \( \hat{p} \) a \( \dot{u}_i \), and then we get

\[
\ddot{u}_2^* + \left[ \frac{\kappa U^*}{\omega - \kappa U} - \kappa^2 \right] \dot{u}_2 = 0,
\]  
(6.6)

With boundary conditions \( \dot{u}_2 = 0 \) on the wall, so for \( x_2 = \pm L \).

This equation, which is referred to as a Rayleigh equation, is used to define the standard Sturm-Liouville problem of eigenvalues \( \omega \). We can solve the problem of proper values for any particular profile of mean velocity \( U(x_2) \) and decide about its stability. Lord Rayleigh already in 1880 proposed a qualitative analysis of equation (6.6), which defines a sufficient condition for the stability of mean velocity profiles.

If we multiply equation (6.6) by \( \dot{u}_2^* \), which is complex conjugate to amplitude \( \dot{u}_2 \) and then perform integration across a channel, we get

\[
\int_{-L}^{L} \dot{u}_2^* \dot{u}_2^* \, dx_2 + \int_{-L}^{L} \left[ \frac{\kappa U^*}{\omega - \kappa U} - \kappa^2 \right] |\dot{u}_2|^2 \, dx_2 = 0.
\]  
(6.7)

Then we integrate by parts

\[
\left[ \dot{u}_2^* \dot{u}_2 \right]_{-L}^{L} - \int_{-L}^{L} |\dot{u}^2| \, dx_2 + \int_{-L}^{L} \left[ \frac{\kappa U^*}{\omega - \kappa U} - \kappa^2 \right] |\dot{u}_2|^2 \, dx_2 = 0.
\]  
(6.8)

The first term on the left side must equal 0, because both \( \dot{u}_2 \) and \( \dot{u}_2^* \) are zero on the walls.

Therefore, in the equation we have a single complex value, which is frequency \( \omega = \omega_r + i \omega_i \). Now we insert this quantity into equation (6.8)

\[
-\int_{-L}^{L} |\dot{u}^2| \, dx_2 + \int_{-L}^{L} \left( \frac{(\omega_r - \kappa U - i \omega_i)\kappa U^*}{\omega - \kappa U} - \kappa^2 \right) |\dot{u}_2|^2 \, dx_2 = 0
\]  
(6.9)

and then we examine the separately imaginary and real part of the equation, both of which must be fulfilled. The imaginary part of equation (6.7) gives us

\[
\omega \kappa \int_{-L}^{L} \frac{U^* |\dot{u}_2|^2}{|\omega - \kappa U|^2} \, dx_2 = 0.
\]  
(6.10)

We now expect that at least one mode exists, which is characterized by \( \omega > 0 \), when exponential growth of waves in time occurs. According to formula (6.10) this is possible only under the
assumption that it is a zero integral. However, this can happen only under the assumption that quantity $U'(x_x)$ changes the sign somewhere in the interval $-L \leq x_x \leq L$.

This fact is expressed in the Rayleigh theorem about the inflection point:

*A necessary condition for linear instability of shear flow of inviscid fluid with the profile of mean velocity $U(x_x)$ is a change of sign $U'(x_x)$ somewhere in the stream.*

We should emphasize that the presence of an inflection point in a profile of mean velocity is a **necessary** condition for inviscid instability of flow due to infinitesimal small perturbations, but it is not a **sufficient** condition!

In the same way, we can also analyze the real part of equation (6.7), and the result is Fjørtoft theorem:

*A necessary condition for instability is that the inequality $U'(U-U_1) < 0$ must apply somewhere in the flow field, when $x_{2s}$ is the point, where $U' = 0$ and $U_1 = U(x_{2s})$.**

This condition is stronger than the Rayleigh condition. Some cases of profiles of mean velocity of flow of inviscid fluid, which are unstable according to Rayleigh criteria, could be stable according to Fjørtoft criteria. An example of velocity profiles during inviscid flow through a channel is shown in Figure 6.5.

![Figure 6.5 – Velocity profiles for inviscid flow in a channel, (a)-(c) stable, (d) possible instability](image)

The profiles on Figure 6.5 (a) and (b) are stable, because $U' > 0$ or $U' < 0$ in the whole area, and therefore the sign does not change. The profile in Figure 6.5 (c) is also stable, because although $U' = 0$ and therefore the Rayleigh condition is fulfilled, according to Fjørtoft the profile is stable, since $U'(U - U_1) \geq 0$ applies in the whole area. Only profile (d) is unstable according to both criteria.

Typical practical examples of flow reporting profiles with inviscid instability include the flow of a jet into a calm environment or wakes behind a body. However, according to these criteria, flow in the channel or the boundary layer without the presence of pressure gradient is stable, because the corresponding velocity profile does not have an inflection point.
5.2.2. Stability of viscous flows

The theory of stability of inviscid flow is relatively well worked out. However, it is a question to what extent the results of this analysis can be applied to flow of real fluid, which is always viscous. The quality effect of viscosity on the results obtained with the help of the theory of inviscid stability was formulated by Osborn Reynolds:

- Flow of inviscid fluid can be unstable, while flow of viscous fluid under the same conditions is stable. Viscosity therefore stabilizes flow.
- Flow of inviscid fluid can be stable, while flow of viscous fluid under the same conditions is unstable. In this case viscosity is the cause of the unstable behavior.

It is apparent from these two somewhat contradictory claims that the results of the stability analysis performed for inviscid fluid needs to be taken with reservations. In the next chapter, we will attempt to show the effect of viscosity on the stability of flow.

We will carry out an investigation of the stability of flow of viscous fluid in a way similar to the investigation for inviscid fluid. Instead of Euler equations, however, our starting point will now be N-S equations. We should again expect planar flow. Let us introduce flow function $\psi$ for perturbation velocity with the help of the following definition:

$$ u'_1 = \partial \psi / \partial x_2, \quad u'_2 = - \partial \psi / \partial x_1. $$  \hspace{1cm} (6.11)

We should anticipate the perturbation flow function in the formulation

$$ \psi = \hat{\psi}(x_2) \exp \left[ i (\kappa x_1 - \omega t) \right]. $$  \hspace{1cm} (6.12)

Boundary conditions now also contain a condition of zero normal and tangent perturbation velocity on the wall, hence the formulation $\hat{\psi} = \hat{\psi}' = 0$ for $x_2 = \pm L$. The resulting equation then has the following formulation

$$ i \nu \left( \hat{\psi}'' - 2\kappa^2 \hat{\psi}'' + \kappa^4 \hat{\psi} \right) + (\kappa U - \omega) \left( \hat{\psi}'' - \kappa^2 \hat{\psi} \right) - \kappa U' \hat{\psi} = 0. $$  \hspace{1cm} (6.13)

This is an Orr-Sommerfeld equation, which was derived already in 1907 and is the equivalent of a Rayleigh equation for viscous flow. It is an ordinary linear differential equation of the 4th order, for which we resolve the problem of eigenvalues $\omega$.

Let us consider Poiseuille flow in a flat channel. The flow occurs as a result of a pressure gradient along the channel. The developed Poiseuille flow has a parabolic profile of mean velocity

$$ U(x_2) = U_{\text{max}} \left(1 - \frac{x_2^2}{L^2} \right). $$  \hspace{1cm} (6.14)

Let us notice that the profile of mean velocity does not have an inflection point, and therefore according to the inviscid theory of stability it is stable. A solution to the problem of stability when considering viscosity is shown in Figure 6.6.
Figure 6.6 – Stability diagram for (a) Poiseuille profile and (b) profile with inflection point

The diagram in Figure 6.6 shows the areas of the positive imaginary part $\omega_i$ and complex frequency $\omega$, which corresponds to unstable development of perturbations, in the level of the parameters of wave number $\kappa$ and Reynolds number $Re = U_{max} L/\nu$. Case (a) in Figure 6.6 corresponds to the parabolic velocity profile of planar Poiseuille flow, and case (b) shows flow characterized by the profile of mean velocity with an inflection point (e.g. a jet flow into a calm fluid). Exceeding the critical Reynolds number $Re_k$ can lead to an increase in perturbations with a corresponding wave number. For Poiseuille flow in a flat channel, the value is $Re_{ka} = 5772$. It generally applies that in relation to a velocity profile that according to the inviscid theory is stable, (a) is an area of viscous instability much smaller and occurs during higher Reynolds number values than in the case of an unstable profile according to inviscid theory (b).

In the example of Poiseuille flow, the double effect of viscosity on its stability can be shown. On one hand, viscosity has a stabilizing effect, because from the stability analysis with consideration of the viscosity, the critical Reynolds number was specified, and if we increase the value of kinematic viscosity $\nu$, it has the result of increasing the critical velocity. On the other hand, viscosity has a destabilizing effect, because it is apparent from an inviscid analysis that the Poiseuille flow is always stable, and by considering viscosity we will reach the final value of the critical Reynolds number.

5.2.3. Types of hydrodynamic instability

In this chapter we will show certain types of instabilities of flow under certain specific conditions. The presented flow fields are the first phase of the transition to a turbulent condition.

5.2.3.1. Kelvin-Helmholtz instability

This is the simplest illustrated case of shear flow, in relation to which loss of stability can occur. Nonetheless, this type of instability very often occurs in practice, such as during surrounding of wings, during tearing away of the boundary layer or in a layered atmosphere. It often occurs at the boundary of two fluids with different physical properties. The situation is illustrated schematically in Figure 6.7. In the stream near the boundary of the occurring perturbation of pressure, overpressure (+) and sub-pressure (-), a deformation of the boundary occurs in the form of periodic vortex structures.
Mathematically, the stability of this free shear layer can be solved in a manner similar to the case of the boundary layer (see above). We can consider the initial velocity profile $U_0(x) = \tanh(x)$. Then the Orr-Sommerfeld equation can be applied, which leads to a stability diagram, which is schematically shown in Figure 6.8. According to this diagram, the free shear layer is unstable for the random Reynolds number value $\text{Re}_d = U_\infty \cdot d / \nu$ ($d$ is the transverse dimension of the shear area). For growing $\text{Re}_d$ the interval of unstable wave numbers $\kappa$ is increased until for sufficiently high Reynolds numbers it reaches its maximum asymptotic values. The rate of the growth of perturbations $\omega_i$ in this case drops with the dropping $\text{Re}_d$. Unlike in the case of a boundary layer, in this case the upper branch of the neutral curve continues to rise. The friction in the free shear layer acts against the loss of stability, and due to perturbations with large wave numbers (small interruptions), the flow is stable.

The actual experimentally obtained view of the free shear layer under the conditions of Kelvin-Helmholtz instability is shown in Figure 6.9.
5.2.3.2. Rayleigh-Bénard instability

During this type of flow, the originally calm fluid is sealed in a dish, and its bottom wall is heated up. The changes in density of the fluid during its heating leads in the gravitational field to the motion of the fluid inside the dish. The parameter that determines the pattern of flow in the dish is the Rayleigh number

\[ \text{Ra} = \frac{\alpha g \Delta T L^3}{\nu}, \]  

(6.15)

where \( \alpha \) is the coefficient of thermal expansion of the fluid, \( \beta \) is the coefficient of thermal diffusion, \( g \) is gravitational acceleration, \( \Delta T \) is the difference between the temperature of the bottom and the lid of the dish, and \( L \) is the vertical dimension of the dish. The Rayleigh number expresses the ratio of gravitational and viscous forces.

During the occurrence of instability, we can observe typical spongy structures – see Figure 6.10. Another phase of Rayleigh-Bénard instability is the occurrence of areas of rotating fluid between the heated and cooled wall, as shown in Figure 3.6 in the chapter regarding deterministic chaos and the Lorenz system.

Figure 6.10 – Rayleigh-Bénard convection – side view

Figure 6.11 shows a stability diagram specifying the area of instability in the level of Rayleigh numbers \( \text{Ra} \) and wave numbers \( \kappa \). The minimum Rayleigh number value for instability \( \text{Ra}_c \) is 1707 and the corresponding wave number, which is decisive for the dimension of occurring cells, is \( \kappa_c = 3,12 \).

Figure 6.11 – Rayleigh-Bénard stability diagram
When looking at flow between walls from the top, we can observe a regular “cell” structure in the form of regular hexagons. The fluid rises in the middle, and on the edges it falls down. A photograph of the visualization from the actual experiment is shown in Figure 6.12.

\[ \text{Figure 6.12 – Rayleigh-Bénard cells} \]

### 5.2.3.3. Taylor-Couette instability

This type of instability occurs during mutual rotation of two aligned cylinders, and the gap between them is filled with viscous fluid. The flow is characterized by a Taylor number, which expresses the ratio of centrifugal and viscous forces. Its definition is

\[
Ta = \frac{\Omega^2 R^3}{\nu^2},
\]

where \( \Omega \) is the angular velocity of rotation of the internal cylinder, \( R \) is the radius of the internal cylinder, \( l \) is the width of the gap between the cylinders, and \( \nu \) is the kinematic viscosity of the fluid. Figure 6.13 shows a diagram of flow in the gap between the cylinders following the loss of stability.

\[ \text{Figure 6.13 – Taylor-Couette instability} \]

The critical value of Taylor number \( Ta_c \) is around 1708, and for higher values the ring vortex structures occur in the gap between the cylinders, and the corresponding wave number is
\[ \kappa = 3.12. \] The value of the wave number rises with the rising Taylor number. The wave length \( \lambda \) on Figure 6.13 is bound with the wave number \( \kappa \) through the relationship: \( \kappa = 2\pi/\lambda \).

### 5.2.3.4. Görtler instability

This type of instability is very similar to the previous case. Instability occurs during flow along the concavely curved wall. Centrifugal and viscous forces again appear in the problem, and their ratio is expressed by Görtler number

\[
\text{Go} = \frac{U_{\infty} \theta (\frac{\theta}{R})^{1/2}}{\nu},
\]

where \( U_{\infty} \) is the velocity of flow beyond the boundary layer, \( \theta \) is the displacement thickness of the boundary layer, \( R \) is the radius of the wall curve and \( \nu \) is the kinematic viscosity of the fluid. The situation is illustrated in Figure 6.14. With this definition of a Görtler number, its critical value necessary for the creation of vortex structures is approximately 0.3.

![Figure 6.14 – Görtler instability](image)

The stability diagram in the space of Görtler numbers \( \text{Go} \) and wave numbers \( \kappa \) of periodic perturbations oriented in a transverse direction is shown in Figure 6.15. The relationship of the wave number and wave length from Figure 6.14 is \( \kappa = 2\pi/\lambda \).

![Figure 6.15 – Görtler stability diagram](image)

### 5.2.3.5. Tollmien-Schlichting instability

Tollmien-Schlichting instability occurs in the boundary layer when achieving the critical value of the Reynolds number
\[ \text{Re}_x = \frac{U_x x_1}{\nu}, \]  

(6.18)

where \( U_x \) is the velocity of flow beyond the boundary layer, \( x_1 \) is the distance from the running edge and \( \nu \) is the kinematic viscosity of the fluid.

The Reynolds number can be defined in two ways, either for a length parameter it takes the distance from the leading edge \( x_1 \) or the thickness of boundary layer \( \delta \). The definitions are equivalent, and only the numerical values of such defined Reynolds numbers are different. Between the length parameters for the laminar boundary layer, the relationship \( \delta \sim x_1 \sqrt{\text{Re}_x} \) applies.

Tollmien-Schlichting waves are presented vortices in the boundary layer focused in a direction perpendicular to the direction of flow, meaning \( x_2 \). The critical value of the Reynolds number is for the case of a laminar boundary layer on a board without the presence of a pressure gradient approximately \( \text{Re}_x = 10^5 \) (this value corresponds to the Reynolds number defined with the help of the thickness of the boundary layer of approximately 500).

We will examine this type of instability in detail in the next paragraph 6.3, which discusses the transition to turbulence.

### 5.2.3.6. Wakes behind bodies

Behind bluff bodies, areas of slowed fluid occur, where repeat flow can occur. At the boundaries between slowed and flow fluid, shear areas are created, in which Kelvin-Helmholtz instability can occur. However, the situation is more complicated, and there is a connection between developments occurring in various parts of the wake through pressure signals. This results in quasi-periodic structures – von Kármán-Bénard vortex street – see Figure 3.12.

The stability diagram for this situation is shown in Figure 6.16 and represents the situation in the level of wave numbers \( \kappa \) and Reynolds numbers \( \text{Re} \) and \( \omega_i \) is an imaginary part of frequency. The Reynolds number is defined in the usual manner based on the transverse dimension of the body. Instability is expressed for Reynolds number in a value greater than the critical value \( \text{Re}_c \), and the corresponding wave number is \( \kappa_c \). For example, for the transversely surrounded round cylinder, the value is \( \text{Re}_c = 44 \), when until this time stable vortices become unstable and a von Kármán-Bénard vortex street is created (see Figure 3.13).

![Figure 6.16 – Stability diagram of a wake](image)

Another case involves jets blown into a calm environment, but this situation is of course opposite the case of a wake – in the area of the jet there is a high velocity, and outside of it the
flow velocity is very low (almost 0). The stability diagram is qualitatively very similar to in the case of the wake in Figure 6.16, and the critical Reynolds number defined with the help of the diameter of the mouth of the nozzle is now approximately 500.

5.2.3.7. Other instabilities

Many various types of instabilities exist. As an example, we can mention the Rayleigh-Taylor instability, which occurs in the presence of two fluids of various densities during the effects of gravitation or Saffman-Taylor instability, which occurs during mixing of two chemically non-reacting fluids, which differ in their viscosity.

It generally applies that flow in the shear area of a random type loses its stability under the condition that the volume forces (inertial, centrifugal, buoyancy, ...) affecting the fluid will have prevalence above viscous forces.

5.3. Transition to turbulence

We can regard the loss of stability of a particular type of flow as the first phase of the transition of the flow to turbulence, which is followed by the scenario of the transition of the shear area to turbulence. It has been shown that this scenario has certain characteristics, which are common for all cases discussed above. We will demonstrate the phenomena connected to the transition of shear layers to turbulence in a “canonic” case of a boundary layer on a flat plate without the presence of a pressure gradient.

In the shear area subjected to the effects of perturbations, two different scenarios of transition to turbulence can occur.

If the amplitude of perturbations caused by external waking is “small” then we can observe more or less regular oscillations related to the spatial structures of wave character, which occur somewhat after the flow from a certain critical point. These waves have a form determined by the very shapes of the particular laminar shear layer. The first phase of their development can be sufficiently accurately described with the help of a linearized model. Regular sections are enlarged, oscillation intensifies, the non-linear character is applied increasingly, and the process ends with the breakdown of regular structures and the occurrence of turbulence. This scenario is usually called a “natural transition to turbulence”.

If the amplitude of external waking is “sufficiently large” then non-linear processes are started directly, and the process of transition to turbulence is completed very quickly. The name of this process “bypass transition” expresses the fact that the linear phase of transition does not occur at all, but is bypassed.

The fact that the behavior of the shear layer is dependent on the amplitude of the perturbations to which it is subjected is related to the non-linear character of the problem. Better than linear stability, the properties of the shear layer are characterized by its receptivity. This property of the shear area shows its development during the effects of actual perturbations, which are characterized by the frequency and amplitude of the final size.

5.3.1. Natural transition to turbulence

The basic scenario of the transition of laminar flow to turbulence is shown schematically in Figure 6.17 for the case of the boundary layer on the board in a view perpendicular to the board. The fluid flows along the board in a laminar manner (position ①) at velocity $U_\infty$, until its motion in a certain location, labeled $\text{Re}_{\text{crit}}$, becomes unstable. Along the flow in the boundary layer, planar perturbations are generated (position ②) known as Tollmien-Schlichting waves, which as a result of secondary instabilities quickly develop into spatial perturbations with a triangular shape position ③). From them, spacious hairpin vortices are created (position ④), which have a tendency to collapse. They further grow and interact with each other, and this creates randomly distributed turbulence spots (position ⑤), and eventually an entirely developed boundary layer is created (position ⑥). Thereby the transition to turbulence is ended.

8 “Canonic” cases are usually defined with the help of very simple boundary or starting conditions, and in literature there is a large amount of comparative material describing them.
The behavior of perturbations in the area of the early stage of the transition (positions 2 to 3) can be described with a linearized model, and the growth of perturbations is exponential in time. The triangular secondary instabilities from Figure 6.17 correspond to classic topology of perturbations, are ranked behind each other and include K-type (Klebanoff). Another option is sharper application of sub-harmonic processes, and then we get alternated checkered topology structures, and we name the scenario H-type (Herbert). We present the visualization of these structures in Figure 6.18. The Figure shows the area of stability loss, and the fluid flows from left to right. Other types of flow structures exist in this area, and their specific form is closely related to the nature of the perturbations that they prompt. In any case, in this phase there is significant stretching of vortical structures, which leads to generation of vorticity.

Along the flow (position 5) boundary amplitudes of these regular perturbations are achieved, and in this phase increasing deviations can be observed from the behavior predicted based on the linear theory. The form of hairpin vortices is still regular, has a periodic character following stress and is schematically shown in Figure 6.19. Figure 6.19 (a) shows initial distortion, then 6.19 (b) development of vortex legs and head and 6.19 (c) evolution of subsidiary vortices and penetration of the legs toward the surface. The legs of hairpin vortices get close to each other, until they mutually interact, which finally leads to the breakdown of the entire structure and creates turbulence spots.
The occurrence of turbulence spots is a random process both in space and in time, but these sections have a clearly defined shape and development. The frequency of the occurrence of turbulence spots along the flow increases, until they fill the entire space – developed turbulent boundary layers are created. Figure 6.20 on the left shows a photo of the surface of a water transition flow, and the right side of the Figure shows a turbulent spot. Position 1 shows the overhanging front part, 2 is the turbulent core, 3 is the peripheral edge, 4 is the area of calming behind the turbulence spot, 5 is a half-angle of expansion of the spot, and 6 is a lateral overhang.

The place where the first turbulence spots appear is often referred to as the onset of transition. This location can be identified both from the recording of signals with the help of a hot wire probe as well as with the help of measuring methods enabling evaluation only of time-related mean values such as pneumatic methods of measuring pressure. In the point at the onset of transition, we can observe the start of a decline of the shape parameter, while the mean value of surface friction here starts to deviate from the values typical for a laminar boundary layer in the transition area the frequency of the occurrence of turbulence spots continues to rise, until these spots evenly fill the entire area. This completes the process of the transition to turbulence. The area of development of turbulence spots is characterized by the occurrence of an intermittent signal during point measuring of velocity. In the time record of the signal, there are sections of a laminar and a turbulent character, depending on how turbulence spots pass through the measuring location. An example of such signal is shown in Figure 6.21. Instead of referring to sections of a “laminar” and “turbulent” signal, it makes sense in this case to refer to sections with “low turbulence” and “increased turbulence”.
In relation to an intermittent signal, the “intermittency coefficient” $\gamma$ is evaluated, which is defined as the ratio of the sum of time sections in which turbulent signal $T_i$ appears to the total period of observation $T$

$$\gamma = \frac{T_i}{T}.$$  \hspace{1cm} (6.19)

Value $\gamma = 0$ means that the signal has an entirely laminar character, $\gamma = 1$ corresponds to the entirely turbulent signal, and the intermittent signal is characterized by the value $0 < \gamma < 1$.

For transition to turbulence of the boundary layer on a flat plate, the size of the linear area represents approximately 75 to 85 percent distance between the leading edge and the onset of transition to turbulence, and so the area of non-linear effects is very short.

If the time mean flow field has a spatial character (for example for arrowhead wing), the linear phase of the process of transition practically does not differ. However, the behavior of unstable waves is different, and they now spread in various direction. According to the linear theory, the direction of the least stable spread of waves is determined by the acceleration or deceleration of the flow in the particular direction. The special characteristic of the instability of the spatial flow is that perturbations with zero frequency (meaning with constant amplitude) become very unstable as soon as the transverse velocity exceeds a certain limit. In experiments these stationary perturbations can be made visible, and they involve more or less regularly placed lines in the flow direction.

Only very little is currently known about the non-linear mechanism of instability for space situations that lead to the transition to turbulence. It is only clear that the area in which the non-linear effects play the dominant role in the case of spatial flow is much greater than in the case of planar flow.

### 5.3.2. Bypass transition to turbulence

Before 1940, experimenters were not capable of identifying Tollmien-Schlichting waves as well as secondary instabilities in a boundary layer. It was assumed that the transition was caused by another type of perturbations and other mechanisms of growth. Morkovin (1969) declared that “we can entirely bypass the mechanism of Tollmien-Schlichting waves”, and this type of transition to turbulence ever since then has been referred to as a “bypass transition”. During bypass transition, the non-linear mechanism of growth of the perturbations is applied directly, which in a classic scenario appears only in the later phases of transition $\circled{5}$ and $\circled{6}$ (see Figure 6.17). It later became apparent that this mechanism truly can be started with large perturbations, such as those resulting from a high level of turbulence. Non-modal growth of perturbations can lead to turbulence during much lower Reynolds numbers, but exponential growth of modal waves is entirely bypassed.

Today it is already clear that all of the aforementioned scenarios include in one or another level the same elements and mechanisms that are related to the occurrence and development of coherent structures in a shear layer. Hairpin vortices play a decisive role also in the case of bypass transition to turbulence as well as in the process of self-maintenance of turbulent flows, as
will be shown later. However, the difference is that these structures occur in these situations entirely irregularly in space and time.

The bypass transition of flow in a shear layer to turbulence occurs in the event of large perturbations affecting flow and is related to the non-linear character of the entire process. In the case of a boundary layer, it can involve perturbations in an incoming stream, which is already turbulized or perturbations penetrating the boundary layer from the wall. In such case it can involve roughness of the surface or its vibrations. There is no exactly defined boundary between “small” perturbations leading to the scenario of natural transition and “large” perturbations that cause a bypass transition to turbulence. If perturbations in an incoming stream are involved, then this limit is approximately 1% for intensity of turbulence. In each case it leads to significant shortening of the area of transition of the boundary layer to turbulence, and the length of the transition area can be significantly shorter in the event of a bypass transition. The area of developed turbulent flow can very closely approach the point of loss of linear stability. Figure 6.22 shows an example of experimentally gained dependence of the Reynolds number of the onset of transition to turbulence in the boundary layer on flat plate $Re_{\theta}$ defined with the help of the displacement boundary layer thickness on the intensity of fluctuations in the velocity in the incoming stream $Tu$. The Reynolds number of loss of linear stability for this case is $Re_{c\theta} = 163$. Let us notice that during high intensity of turbulence of the incoming stream, the process of transition can begin even before the location of loss of stability according to the linear theory. This is because the process of loss of stability is also in fact non-linear, and with the growing amplitude of perturbations, the actual stability limit decreases.

![Figure 6.22 – Turbulence onset Reynolds number function of inlet flow intensity of turbulence](image)

In the process of bypass transition to turbulence, a role is played not only by the intensity of perturbations, but also by their character and structure. Especially important are the characteristic measuring scales of perturbations and incoming stream, which are decisive for their suppression.

It generally applies that if we are able to ensure that amplitudes of perturbations in shear flows are extremely low, the transition to turbulence can be significantly postponed.

For modeling the shortened transition of flow to turbulence, the linear theory of stability is unusable. So far no general theory exists for modeling this type of transition to turbulence, and only phenomenological models based on experimental data have been created.
6. Developed turbulence

In inviscid fluid, Kelvin theory applies, which states that circulation of a closed material curve is constant. According to this theory, if a curve is stretched as a result of situations during flow, then for maintaining the circulation value, there must be an increase in the rate of the circulation flow in the vicinity of the curve. There is a well-known analogy about a figure skater doing a pirouette. If he has his hands far away from his body, then his moment of inertia is large and the rotation velocity is low. If the hands move as closely as possible to the rotation axis (upward or close to the body), then his moment of inertia is reduced, and while maintaining rotation energy, the rotation velocity is increased. This mechanism for reducing the size of vortex structures and increasing vorticity is applied during the cascade transfer of energy. Vortices are stretched, and this reduces their transverse dimensions, and the values of velocity gradients increase. The mechanism of generating vorticity through stretching of vortices will be described in detail in Chapter 10.2.4.

Let us imagine the development of vortex structures in an area of flow fluid, where in a certain time large-scale vortex structures are generated as a result of external forces. The vortex fibers or pipes stretch, become deformed and variously sway. This creates a randomly arranged set of increasingly smaller vortex structures. If the dimensions of these vortex structures is sufficiently large, the viscosity of the fluid does not have a fundamental effect on this process. Local strengthening of vorticity occurs. If during this process we monitor a certain sufficiently large limited area, then as a result of the conservation of kinetic energy (we can overlook dissipation) the intensity of the fluctuations of the velocity field during the process of reducing scales is the same. From that it is clear the vorticity contained in a unit of fluid volume cannot grow very much.

However, as soon as the dimension of the vortices falls to below a certain limit, the viscosity gradually becomes more significant. The term containing the viscosity and the second spatial derivative in the N-S equations grows more quickly than the term containing derivative only of the first order. This leads to a significant decrease of the Reynolds number related to the dimensions of the structures, and its value is already too low. The effect of viscosity thus stops the process of local growth of vorticity and reduction of the size of vortex structures. Diffusion of the vorticity occurs as a result of the growth of the diffusion term in the equation for vorticity. This process occurs in the area of scales that approach the Kolmogorov scale. These are the smallest observable vortex structures, and they further dissipate. As soon as these structures occur in turbulent flow, the process of its development ends, and we consider turbulence to be “developed”. The field of velocity has a random character and contains a whole range of scales. This creates a continuous energy flow from the largest scales to the smallest, and the smallest then dissipate. Stretching of vortices plays an important role in this process.

However, there are certain categories of processes in which stretching of vortices does not occur. These include planar, two-dimensional flows. In such flows, the vortex lines must obviously be perpendicular in relation to the flow level, and they cannot be stretched. Therefore, the mechanism described above for the creation of smaller vortex structures is not even possible, and in the end result even developed turbulent flows cannot form. The vorticity is only carried unchanged by stream. In relation to symmetrical flows, the vortex lines can have circular shape, and their radius determines the intensity of the vorticity. Therefore, stretching of vortex fibers can occur, but such case is very special. If in such nominally two-dimensional flow spatial motion of fluid is enabled, then even under such conditions classic turbulent flow can occur. However, situations exist when this is not possible, such as flows in extremely thin layers.

So far we have examined turbulent flows without the presence of a limiting solid border. Solid borders are often an important source of turbulence, and in the end all vorticity occurs at borders, and inside fluid it is then intensified with the help of a stretching mechanism. The behavior of vortices and turbulence within shear areas significantly differs from behavior at a great distance from the surface. The value of an effective Reynolds number is relatively low here, and both the production and the dissipation of turbulent energy are very intense. This is caused by the intensive shear character of the area and other factors. However, the process of increasing
vorticity occurs here with the help of the same mechanisms, and the stretching of vortices plays an important role. As a result of the low Reynolds number, the range of sizes of vortex structures is greatly limited. The Reynolds number also grows from the wall, and the structure of flow approaches free turbulence, although the effect of the limiting wall can be apparent to a significant distance. If the distance of the wall is greater than the value of correlation of the length, then the effect of the wall can be overlooked.

6.1. Statistical description of turbulence

Years of experience have shown that the application of Navier-Stokes equations to cases of laminar flows yields reliable results, which correspond well with the results obtained from experiments. However, for the examination of turbulent flows, we must take a different approach, because the quantities characterizing flows are basically random. This means that any specific values of flow characteristics coming from theory are incorrect in comparison with a specific experiment. From the theory, it is possible with the help of mathematical modeling to predict only the likelihood that a certain event will occur. Quantities such as the velocity of flow in a particular point need to be approached as random quantities, and it is also necessary to make use of appropriate mathematical tools.

Statistical tools present exactly defined results only for random processes, which are homogeneous in space. In practice this requirement generally has not been fulfilled, so let us replace it with a more moderate requirement of a stationary state of the process in the sense of statistical quantities.

6.1.1. Spectral characteristics

Spectra are basic statistical characteristics of signals of a random character, which contain in them information about the sizes of structures. The definition results and methods of calculation are specified in the addenda.

For monitoring spatial spectra, the term “wave number” has been introduced, which is defined by the relationship

$$\kappa = \frac{2\pi}{\ell},$$

where \( \ell \) is the vector of wave length. The wave number and the wave length are a form of frequency and period time, with the difference that they involve vector oriented in space.

6.1.1.1. Velocity spectrum

The velocity spectrum \( \Phi_{ij}(\kappa) \) is a 2\(^{nd}\) order tensor, which tends to be referred to as a “velocity spectrum tensor”. It is defined as a Fourier transformation of a spatial two-point correlation of velocity \( R_{ij}(r) \); that is implemented in a manner similar to auto correlation in time. The two-point correlation is a second-order tensor \( R_{ij} \)

$$R_{ij}(r) = u_i'(x + r, t)u_j'(x, t),$$

where \( x \) is the vector of the position of the particular point, \( r \) is the sliding vector and \( u_i, u_j \) are parts of the vector of velocity into corresponding directions. In homogeneous turbulence, the mutual relationship between the velocity spectrum and correlation can be expressed by the equations

$$\Phi_{ij}(\kappa) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ij}(r)e^{-i\kappa \cdot r} dr,$$

$$R_{ij}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}(\kappa)e^{i\kappa \cdot r} d\kappa,$$

where \( \kappa = (\kappa_1, \kappa_2, \kappa_3)^T \) is the vector of wave numbers (continuous), \( r = (r_1, r_2, r_3)^T \) is the vector of spatial sliding.
From a physical point of view, the velocity spectrum tensor represents the density of Reynolds tension in the space of the wave numbers. For example, for \( r = 0 \) from equation (7.4) the following relationship is immediately apparent:

\[
R_y(0) = \overline{u_i' u_j'} = \int \int \int \Phi_y(\kappa) d\kappa.
\]  

(7.5)

The velocity spectrum tensor is a complex quantity, which is positively semi-definite. The information about it can be divided into several parts. The first piece of information is found in indexes (i a j), which identify the direction of velocity in physical space. Other information is found in the vector of wave numbers \( \kappa \). Its direction \( \kappa/|\kappa| \) indicates the direction of the Fourier mode in physical space and finally its size is determined by the corresponding spatial scale \( \ell = 2\pi/|\kappa| \).

The tensor also contains information about velocity derivatives

\[
\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_l} = \int \int \int \kappa_i \kappa_j \Phi_y(\kappa) d\kappa,
\]  

(7.6)

and therefore with its help we can also express the rate of dissipation

\[
\varepsilon = \int \int \int 2\nu \kappa^2 \frac{1}{2} \Phi_y(\kappa) d\kappa.
\]  

(7.7)

The correlation function, like a spectrum, is generally a function in time, and in the specified expressions this dependency is not labeled in order to make it clearer.

### 6.1.1.2. One-dimensional spectrum

Almost all experimental data from the research of turbulent flows originates from measuring using hot-wire probes, which are located in a fixed point in space. For determining the spatial correlations, the Taylor hypothesis is used (see 7.1.3), evaluation is performed of longitudinal and transverse correlation coefficients \( f(\vec{r}) \) and \( g(\vec{r}) \), which are defined for motion in the direction of axis \( x_i \):

\[
f(\vec{r}) = R_{11}(\vec{r}, \vec{r}) / \bar{u}_i^2 = u_i'(x + \vec{e}_i \vec{r}, t) u_i'(x, t) / \bar{u}_i^2,
\]

\[
g(\vec{r}) = R_{22}(\vec{r}, \vec{r}) / \bar{u}_j^2 = u_j'(x + \vec{e}_j \vec{r}, t) u_j'(x, t) / \bar{u}_j^2,
\]  

(7.8)

Where \( \vec{e}_i \) is the unit vector in the direction of \( x_i \). The isotropic homogeneous turbulence is fully characterized by two non-dimensional correlation coefficients, and for other parts of tensor \( R_y \), the following also applies: \( R_{33} = R_{22} \) and \( R_y = 0 \) during \( i \neq j \).

The one-dimensional spectrum is then defined by the Fourier transformation of the correlation function

\[
E_y(\kappa_i) = \frac{1}{\pi} \int_{-\infty}^{\infty} R_y(\vec{r}, \vec{r}) e^{i\kappa_i \vec{r}} d\vec{r},
\]  

(7.9)

where \( \kappa_i \) is the wave number in the direction of axis \( x_i \), \( j \) equals 1 or 2.

The one-dimensional spectrum is related to the velocity spectrum tensor

\[
E_{11}(\kappa_i) = 2 \int \int \Phi_1(\kappa) d\kappa_2 d\kappa_3.
\]  

(7.10)

Let us notice that the resulting one-dimensional spectrum \( E_{11} \) contains contributions from all wave numbers \( \kappa \) at the level \( e_i \cdot \kappa = \kappa_i \) (projection to this level). However, the resulting amplitude \( |\kappa| \) can be much greater than \( \kappa_i \).

We can simplify the practical calculation of the one-dimensional spectrum \( E_y \) based on the knowledge that \( R_y \) is always an even function of \( \kappa_i \). Then the following obviously applies
\[ E_y(\kappa) = \frac{2}{\pi} \int_0^\infty R_y(e, r_1) \cos(\kappa r_1) \, dr_1 \]  
\hspace{1cm} (7.11)

and inverse transformation
\[ R_y(e, r_1) = \int_0^\infty E_y(\kappa, r_1) \cos(\kappa r_1) \, d\kappa_1 . \]  
\hspace{1cm} (7.12)

In the case of isotropic turbulence, the one-dimensional spectrum is tied with the energy spectrum \( E(\kappa) \), which will be defined in the following chapter.

### 6.1.1.3. Energy and dissipation spectrum

The velocity spectrum tensor contains a significant amount of information, which would be somewhat difficult to use practically. Therefore, for greater clarity, the term “energy spectrum” is being used, which represents a somewhat reduced value.

The energy spectrum is formed from the velocity spectrum tensor by suppressing information about the directions of velocity in physical space and information about the directions of wave numbers in Fourier space. Mathematically, this means the calculation of the path of a 2\textsuperscript{nd} order tensor and integration along the surface of the sphere in Fourier space with a radius of \( \kappa = |\kappa| \) with the mean at the beginning:

\[ E(\kappa) = \iint \frac{1}{2} \Phi_y(\kappa) dS(\kappa) . \]  
\hspace{1cm} (7.13)

Energy spectrum \( E(\kappa) \) is a scalar function of scalar variable \( \kappa \). It is apparent from the definition of the energy spectrum that integration of \( E(\kappa) \) across all \( \kappa \) is equivalent to integration \( \frac{1}{2} \Phi_y(\kappa) \) across all \( \kappa \). Therefore, for kinetic energy, the following relationship applies

\[ k = \int_0^\infty E(\kappa) \, d\kappa . \]  
\hspace{1cm} (7.14)

Energy spectrum \( E(\kappa) \) therefore represents the division of kinetic energy into individual wave numbers \( \kappa \).

We can take a similar approach with regard to the rate of dissipation of turbulence \( \varepsilon \) (we of course have in mind the mean value of this number, and for practical reasons we will omit the bar). The following relationship applies for it:

\[ \varepsilon = \int_0^\infty 2\nu \kappa^2 E(\kappa) \, d\kappa . \]  
\hspace{1cm} (7.15)

The expression inside the integral is a dissipative spectrum \( D(\kappa) \)

\[ D(\kappa) = 2\nu \kappa^2 E(\kappa) . \]  
\hspace{1cm} (7.16)

With the help of the dissipative spectrum, we can, for example, express the dissipation in the interval of wave numbers \( (0, \kappa_i) \) as \( \varepsilon(\kappa_i) = \int_0^{\kappa_i} D(\kappa) \, d\kappa \).

### 6.1.2. Taylor hypothesis

The spatial velocity spectrum tensor \( \Phi_y(\kappa) \) is defined as a Fourier transformation of two-point correlation of velocity \( R_y(r) \). Full measuring of the entire tensor is very demanding and practically is not carried out. With an anemometer with a single sensor, it is possible to measure two-point correlation along the section in the direction of mean flow velocity. The equipment is called a “moving hot wire”, and the probe moves towards the turbulent flow field (meaning towards the flow after deduction of the mean velocity) at constant velocity \( U \) in the direction of \( \mathbf{e}_i \) (unit vector \( e_i \)). For the instantaneous position of probe \( \xi(t) \) the following relationship applies

\[ \xi(t) = x_0 + e_i Ut , \]  
\hspace{1cm} (7.17)
where \( x_0 \) is the starting position of the probe. The velocity indicated by the probe will be ‘\( u \)’:

\[
'\mathbf{u}(t) = \mathbf{u}(\xi(t)),t) - eU.
\] (7.18)

The auto-correlation determined from the data indicated by the probe will be

\[
'\mathcal{R}_{ij}(\tau) = \mathcal{R}_{ij}(t) = \int \int \left[ 'u_i(t) - 'u_i(t) \right] \left[ 'u_j(t+\tau) - 'u_j(t+\tau) \right] du dt.
\] (7.19)

where \( r_i = U\tau \) is the distance by which the probe relocates behind time \( \tau \). For the homogeneous turbulent velocity fields, we reach a limit when the velocity of the motion of the probe goes beyond all boundaries

\[
'\mathcal{R}_{ij}(\tau) = u_i(\xi(t),t)u_j(\xi(t+\tau),t+\tau) = u_i(\xi(t),t)u_j(\xi(t+e\xi),t+e\xi/U).
\] (7.20)

We can see that the measured auto-correlation of the function in the time area crosses in a spatial auto-correlation function in point \( (\xi,0) \).

In a practical case, when the velocity of the motion of the probe is bounded, the time auto-correlation function is ascertained through approximation of the spatial auto-correlation function, and the better it is, the higher the velocity of the motion of probe \( U \).

An easier variant of this method involves the use of a solid probe in space, in which case it obviously applies that \( eU = \overline{u} = -\overline{e\xi} \) and also \( r_i = -\overline{u} \). This method gives good results for low intensity of turbulence, when \( u' \ll \overline{u} \).

The replacement of a spatial correlation with a time correlation is referred to in literature as “Taylor hypothesis” or the “method of approximation of frozen turbulence”.

The success of this approximation depends both on the properties of flow and on the statistical quantities that will be evaluated. It was verified, for example, that in the case of grid turbulence for low intensity of turbulence, this method is satisfactory also for higher level characteristics. However, for free shear flows, this method generally cannot be used.

### 6.1.3. Structure functions

For a detailed analysis of turbulent signals, structure functions are often used. Structure functions are statistical characteristics of growth of a particular function, which have expression capabilities during the study of correlations. They can be used to assess the stationary and intermittent character of a process. They can also be used for the study of non-stationary processes with stationary growth, both of a fractal and multi-fractal character.

The structure function of the \( n \)-th order of random scalar field \( q \) is defined by the following relationship

\[
S_n(r) = \int \left[ \Delta^n(q(x+r) - q(x)) \right] dx.
\] (7.21)

A structure function can be counted with the help of probability density function of growth of the variable \( q \), which is defined by the relationship \( \Delta(q) = q(x+r) - q(x) \). Then it is defined with help of the \( n \)-th order moment of the probability density function

\[
S_n(r) = \int \Delta^n(q) f(\Delta(q))d(\Delta(q)).
\] (7.22)

A general characteristic of the structure functions is the fact that the number of points in the signal for their determination with particular accuracy increases with order \( n \). A structure function of higher orders is very difficult to determine with sufficient accuracy.

The purpose of structure functions is to measure the intensity of fluctuations depending on a scale directly in the space of coordinates.
6.2. Turbulence scales

Turbulent flows consist of vortex structures of various sizes, and they can be characterized using turbulence scales.

6.2.1. Cascade of scales

Let us consider the developed turbulent flows with the velocity characteristic $U$ and the length scale $L$. The Reynolds number $\text{Re} = \frac{UL}{\nu}$ should acquire large values.

The idea of an energy cascade was first formulated by L.F. Richardson in 1922. The basis of this idea is the assumption that developed turbulent flow consists of a large number of vortices of various scales. A vortex with size $l$ corresponds to characteristic velocity $u(l)$ and the time scale $\tau(l) = l/u(l)$. An exact definition of a vortex is somewhat problematic, and so for our purposes we will understand it as an area with a dimension of $l$ that reports a certain level of coherence. The area belonging to a vortex of a certain size can also contain other smaller vortices. For representation of the sequence of vortex scales, Richardson proposes Fourier space, where the size of vortex $l$ is characterized by wave number $\kappa = 1/l$.

Vortices of the largest dimensions are created with the help of mechanisms of instability of shear areas, which were described in Chapter 6. They are characterized by the length scale $l_0$, which is of order $L$ and the corresponding velocity $u(l)$ is of order of a symmetrical deviation of velocity $u' = \sqrt{2/3k}$ and is comparable with $U$. Their dimension is based on the dimensions of the flow area, and these vortices are livened up with energy from the main flow. The Reynolds number of these vortices $\text{Re}_0 = u_0 l_0 / \nu$ is therefore large and comparable with Re, and therefore the effect of viscosity is insignificantly small.

Richardson assumes that the largest vortices are unstable, disintegrate and transfer energy to somewhat smaller vortices. These smaller vortices go through a similar process of disintegration and transfer their energy to even smaller vortices. In the mechanism of disintegration, viscosity plays an insignificant role. This “energy cascade” continues until the Reynolds number of the vortex is small enough that the vortex motion becomes a stable effect of viscous forces and the direct dissipation of the kinetic energy of the vortex occurs. Richardson characterized this process in the form of a son, in which he paraphrased a sonnet from J. Twist about similarities in biology, particularly about fleas that have their own fleas, etc.:

“Big whorls have little whorls
That feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity – in the molecular sense.”

In this model, viscosity figures only at the very end of the cascade of processes. However, the rate of dissipation $\varepsilon$ is determined by the first sequence of processes, which is the transformation of the energy of the largest vortices. The energy of these vortices is of order $u_0^2$ and time scale $\tau_0 = l_0 / u_0$, and the rate of the transfer of energy is characterized by the scale $u_0^2/\tau_0 = u_0^3/l_0$. This size of the scale of the rate of dissipation was experimentally confirmed for large Reynolds numbers and does not depend on viscosity.

6.2.2. Definition of turbulence scales

The complex structure of turbulent flows can be quantified with the help of characteristic scales of turbulence, which we are presenting here.

In Chapter 7.1.1.2 regarding a one-dimensional spectrum of velocity, non-dimensional longitudinal and transverse correlation coefficients $f(r)$ and $g(r)$ were defined by relationship (7.8), and now we will define characteristic scales for these coefficients.

---

9 *whorls* are “spirals” in the sense of “vortices”
The integral scale characterizes the mean dimension of energy vortices. Two integral scales can be defined, a “longitudinal integral scale” $L_{11}$ and a “transverse integral scale” $L_{22}$:

$$L_{11} = \int_0^\infty f(r_i) dr_i,$$
$$L_{22} = \int_0^\infty g(r_i) dr_i.$$  \hfill (7.23)

In isotropic turbulence, the transverse integral scale has half the size of a longitudinal scale.

Another scale, which is often defined, is the Taylor microscale. The Taylor microscale does not have a clear physical meaning. It characterizes mean spatial gradients of velocity as well as vortices that still contribute a little to the process of dissipation, but only have left very little turbulent energy. In any case, it is a scale between the Kolmogorov scale and the sizes of energy vortices, which are clearly defined. Mathematically the Taylor microscale is defined with the help of a second derivative of a non-dimensional correlation coefficient for $r_i = 0$:

$$\lambda_f = \left[ -\frac{1}{2} \left( \frac{\partial^2 f}{\partial r_i^2} \right)_{r_i=0} \right]^{\frac{1}{2}},$$
$$\lambda_g = \left[ -\frac{1}{2} \left( \frac{\partial^2 g}{\partial r_i^2} \right)_{r_i=0} \right]^{\frac{1}{2}}.  \hfill (7.24)$$

The size of the Taylor microscale can be determined with the help of an osculating parabola interwoven with the peak of the correlation function – see Figure 7.1.

![Figure 7.1 – Turbulence scales](image)

The following relationship applies in isotropic turbulence between a longitudinal and transverse scale

$$\lambda_g = \lambda_f / \sqrt{2}.$$  \hfill (7.25)

It is also possible to define the relationship between the rate of dissipation $\varepsilon$ and the Taylor microscale

$$\varepsilon = 15 n \overline{u^2} / \lambda_g^2,$$  \hfill (7.26)

where $u'$ is part of the fluctuation velocity in a random direction.
The Reynolds number based on the Taylor microscale is often used to demonstrate characteristics of isotropic turbulence

\[ \text{Re}_\lambda \equiv \frac{\sqrt{\nu^2} \lambda_\lambda}{\nu}. \]  \hfill (7.27)

Therefore for this Reynolds number the following relationship applies

\[ \text{Re}_\lambda = \sqrt{(20/3)\text{Re}_L}. \]  \hfill (7.28)

The Kolmogorov scale will be more exactly defined in the chapter on Kolmogorov theory, but for now it will be sufficient to state that it involves a characteristic size of the smallest vortices, in relation to which dissipation processes are dominant. Figure 7.1 shows its size \( \eta \), and the value of the coefficient of correlation within it does not deviate significantly from 1, which means that the sections of these sizes move more or less like compact bodies.

We can further define \( L \) as a scale characterizing the largest vortices with the relationship

\[ 32/L_k \equiv \epsilon \]  \hfill (7.29)

where \( \epsilon \) is the rate of dissipation and \( k = (1/2)\langle u^2 \rangle \) is the measured kinetic energy. We can also define the Reynolds number of turbulence

\[ \text{Re}_L \equiv \frac{k^{3/2}L}{\nu} = \frac{k^2}{\epsilon \nu}. \]  \hfill (7.30)

The relationships between the scales can be expressed as follows

\[ \frac{\lambda_\lambda}{L} = \sqrt{10 \text{Re}_L^{-1/2}}; \]  \hfill (7.31)

\[ \frac{\eta}{L} = \text{Re}_L^{-1/3}; \]  \hfill (7.32)

\[ \frac{\lambda_\lambda}{L} = \sqrt{10 \epsilon^{2/3}L^{2/3}}. \]  \hfill (7.33)

Generally correlation coefficients and scales are time functions, but if we consider a stationary case, then we do not have to consider this dependency.

### 6.2.3. Fractal structure of scales

Fractal structures occur in many physical phenomena, such as in turbulence or chaotic dynamical systems. A central role in the characteristics of fractals is played by their dimensions, as was already stated in Chapter 3.2.1.

Geometric objects in the space of dimension \( d \) have their topological dimension \( d_f \). The fractal dimension of a particular geometric object \( d_f \) can be defined with the help of a method of covering the object with the smallest possible number of hypercubes \( N(\epsilon) \) with the edge of length \( \epsilon \)

\[ N(\epsilon) \sim \epsilon^{-d_f} \text{ for } \epsilon \to 0. \]  \hfill (7.34)

We understand a hypercube as a symmetrical, regular object characterized by the length of the edge for the particular dimension – a square for \( d_f = 2 \), a cube for \( d_f = 3 \). This definition characterizes the Hausdorff method of determining fractal dimensions. For smooth objects with continuous derivatives, it usually applies that \( d_f = d_r \), but for objects of a fractal nature the following applies: \( d_f \geq d_r \). An example is the random trajectory on a level, which has \( d_f = 1 \) and \( d_r = 2 \).

The laws of scales in nature cannot be characterized by a single geometric parameter. However, we can consider the property of scales of a certain density \( \mu(x) \) on an object (most often the density of probability). Then we can define the metric

\[ p_\epsilon(x) = \int_{B(x,\epsilon)} \mu(y) \, dy, \]  \hfill (7.35)

where \( B(x,\epsilon) \) is a hypercube with edge \( \epsilon \) with the center in the point of object \( x \).
The general metric \( p_\varepsilon (x) \) grows with exponent \( \alpha \), which is a function of the particular location \( x \)

\[
p_\varepsilon (x) \sim \varepsilon^\alpha ,
\]

(7.36)

It generally applies that \( \alpha \neq d_f \). Therefore, this object can be understood as a super position of various fractals

\[
F(\alpha) = \{ x : p_\varepsilon (x) \sim \varepsilon^\alpha , \varepsilon \to 0 \},
\]

(7.37)

and each of them is characterized by a different exponent \( \alpha \).

This object is then called multifractal. The fluctuations of exponent \( \alpha \) can be characterized by division of probabilities. Such structures are typical for turbulent flow of fluid. We can understand them as objects in relation to which fractal dimensions of turbulence change from location to location and change in time.

The fractal structure of turbulence is expressed both directly in the turbulent flow field through the occurrence of various structures in space and time and at the boundary of the area of turbulently flow fluid. This boundary then also has a fractal character.

A direct connection between the energy cascade and the fractal character of the distribution of parameters in space and time exist. Some phenomenological theoretical models are connected with the spectrum of the cascade of kinetic energy in the form \( \kappa^{-c} \). Then it can be concluded that the fractal dimension of the space in which coherent vortices occur is equal to \( C+1 \). In the case of developed turbulence characterized by the Kolmogorov energy spectrum, which will be described further, the structure of the flow field can be described by fractal dimension \( 5/3+1=8/3=2.67 \). This result has been verified by many experiments, both physical and numerical.

6.3. Kolmogorov theory

In 1941 Kolmogorov published a fundamental article that gives a mathematical apparatus to Richardson’s idea of energy cascade. Kolmogorov theory is based on three hypotheses: a hypothesis of local isotropy and the first and second similarity hypotheses. This theory tends to be referred to in literature as K41.

The hypothesis of local isotropy relates to small-scale vortices. The largest vortices have approximately the dimension of shear area \( \mathcal{E} \), and the topology of these vortices is anisotropic, which can be attributed to specific boundary conditions (often very regular). The mean size of energy vortices is somewhat smaller, and we will refer to it as \( \ell_0 \). The more or less chaotic process of the transfer of energy towards small scales is leading to a gradual increase in the isotropy of smaller scales. This is the basis for Kolmogorov hypothesis of local isotropy: During sufficiently high Reynolds numbers, the motions of small scales \( \ell \ll \ell_0 \) are statistically isotropic.

The scale \( \ell_{EI} \) is the boundary between small isotropic vortices and large non-isotropic vortices. For a better idea, let us consider \( \ell_{EI} \approx 1/6 \ell_0 \). In the area of occurrence of small isotropic vortices \( \ell < \ell_{EI} \) two mechanisms of energy transfer dominate: transfers from large scales to small scales and viscous dissipation. The parameters that manage these processes are the rate of energy transfer from large scales to small \( \mathcal{E}_{EI} \) and kinematic viscosity \( \nu \). In a settled state, the rate of dissipation \( \varepsilon \)\textsuperscript{10} is in balance with the rate of production: \( \varepsilon = \mathcal{E}_{EI} \). It is apparent from this that the statistical universal state of small scales is determined by viscosity \( \nu \) and the rate of the transfer of energy from the area of large scales \( \mathcal{E}_{EI} \). This outcome formulates Kolmogorov first similarity hypothesis, which says that in a turbulent flow with sufficiently high Reynolds number, the statistics of motions of small scales (\( \ell < \ell_{EI} \)) have a universal formulation and depend only on scale \( \ell \), viscosity \( \nu \) and dissipation rate \( \varepsilon \).

\textsuperscript{10} To be understood as a mean time value.
It is apparent from this that the energy spectrum \( E(\kappa) \) has a universal formulation and depends only on \( \nu \) and \( \varepsilon \). If we use these quantities for expressing the energy spectrum, then from the direct dimension analysis it is apparent that this dependency must have the formulation

\[
E(\kappa) = (\nu^5)^{1/4} \varphi(\kappa\eta),
\]

where \( \varphi(\kappa\eta) \) is the Kolmogorov spectrum function. However, for the purposes of the dimension analysis, we can also use \( \varepsilon \) and \( \kappa \), and then we get

\[
E(\kappa) = \varepsilon^{2/3} \kappa^{-5/3} \Psi(\kappa\eta),
\]

where \( \Psi(\kappa\eta) \) is the compensated Kolmogorov spectrum function.

The area of scales \( l < \ell_{el} \) is usually referred to as the universal equilibrium range. In this area, the scales \( l/u(\ell) \) are small in comparison with \( \ell_0/u_0 \), and small vortices can quickly adapt in order to conserve dynamic equilibrium with the rate of energy transfer \( \mathcal{S}_{el} \), which is determined by large vortices \( u(\ell) \) is the typical value of fluctuations in velocity for perturbation of scales \( l \) and \( u_0 \) and then for \( \ell_0 \).

From the dimensional analysis, it is possible to specify clearly (except the non-dimensional constant) the values of resulting Kolmogorov scales. The relevant quantities are only the rate of dissipation \( \varepsilon \left[ m^2/s^3 \right] \) and kinematic viscosity \( \nu \left[ m^2/s \right] \), length, velocity and time-based Kolmogorov scale \( \eta \left[ m \right] \), \( u_\eta \left[ m/s \right] \) and \( \tau_\eta \left[ s \right] \) and then we can define the following relationships:

\[
\eta \equiv \left( \frac{\nu^3}{\varepsilon} \right)^{1/4},
\]

\[
u_{\eta} \equiv \left( \frac{\varepsilon \nu}{\eta^3} \right)^{1/4},
\]

\[
\tau_{\eta} \equiv \left( \frac{\nu}{\eta^3} \right)^{1/2}.
\]

Two identities are apparent from these definitions. First the Reynolds number based on Kolmogorov parameters is a unit figure: \( \eta u_\eta / \nu = 1 \). This fact is in accordance with the claim that the cascade transfer continues in the direction towards continuously smaller scales, until the Reynolds number is so small that it enables dissipative processes. Also from the relationships (7.40) and (7.42) we can express the rate of dissipation

\[
\varepsilon = \frac{\nu^3}{\eta^3} = \frac{u_\eta^4}{\nu} = \frac{\nu}{\tau_\eta},
\]

From which specific characteristics for the velocity gradient of dissipating vortices are apparent:

\[
\left( u_\eta / \eta \right) = 1 / \tau_\eta.
\]

We can also introduce non-dimensional coordinates and non-dimensional speeds with the use of corresponding Kolmogorov scales, in which the following applies for non-dimensional coordinate \( y \) and velocity \( w \):

\[
y = x / \eta, \quad w = u / u_\eta.
\]

In the area of small scales, according to the hypothesis above all turbulent flow fields are statistically similar, and following the transformation with the help of Kolmogorov scales, they are then identical in a statistical sense.

If we consider that \( \varepsilon \sim u_\eta^3 / \ell_0 \), then we can express the relationship of the sizes of the smallest and largest scales in the particular turbulent flow

\[
\frac{\eta}{\ell_0} \sim \text{Re}^{-3/4}, \quad \frac{u_\eta}{u_0} \sim \text{Re}^{-1/4}, \quad \frac{\tau_\eta}{\tau_0} \sim \text{Re}^{-1/2}.
\]

Kolmogorov second similarity hypothesis states that in every turbulent flow during a very high Reynolds number approaching infinity, the motions of turbulent scales \( \ell \) are such that
If we introduce the scale $l_0^I$ (its size is approximately $60\eta$) it is such that Kolmogorov second hypothesis applies in the extent $l_\eta > l > l_0^I$. Such defined scale divides the area of universal equilibrium into two subranges: the inertial subrange, where $l_\eta > l > l_0^I$ and the dissipation range, where $l_\eta > l$. The dissipation range is the only range where there is a significant effect of viscosity, and only Kolmogorov first detailed hypothesis applies for it.

The last area $E > l > l_\eta$ remains, which is the area of the largest vortices and which is usually referred to as the energy-containing range. The division into individual areas is labeled on Figure 7.2, where the axis of scales is logarithmic.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{scales.png}
\caption{Scales of turbulence ranges after Kolmogorov}
\end{figure}

In the inertial subrange, the effect of the viscosity is insignificant, and the coefficient $\kappa\eta \ll 1$ is insignificantly small. Therefore, in the inertial area, the compensated Kolmogorov spectrum is defined (7.39) approximately by constant $\Psi(\kappa\eta) = C$ and the energy spectrum can be expressed by the relationship

$$E(\kappa) = Ce^{2/3}\kappa^{-5/3},$$

where $\kappa$ is the wave number and $C$ is the universal constant ($\approx 1.5$).

### 6.3.1. Isotropic turbulence spectra

The topology of isotropic turbulent flow can be described in a statistical sense with the help of the energy spectrum $E(\kappa)$, which is basically a performance spectral density of velocity fluctuations. This spectrum shows us the distribution of kinetic energy in individual wave numbers. Figure 7.3 shows a model energy spectrum, which shows the typical form of an energy spectrum obtained from an experiment. In the graph, non-dimensional variables are used, which are defined with the help of Kolmogorov scales. The energy spectrum in log-log coordinates is characterized by an inclination of 2 in the energy area and an inclination of $-5/3$ in the inertial subrange. In the area of dissipation, there is quicker suppression of the spectrum. The inclination of the spectrum in the energy area is not entirely clear, and its actual value depends on specific boundary conditions. The specified value expects isotropy.
The form of the energy spectrum in Figure 7.3 is universal to a great extent, and only the size of the inertial subrange is a function of a Reynolds number. We are showing this dependency in two examples, in which somewhat different scaling of variables was shown. In Figure 7.4 (a) an expression of the integral scale has been used characterizing the largest vortices, and in Figure 7.4 (b) the Kolmogorov scale is used like in Figure 7.3. The variable is a Reynolds number related in the Taylor microscale.

It is clear from Figure 7.4 that the definition of variables with the help of the dimension of large vortices causes a universal representation of the energy area, while the use of Kolmogorov scales causes universal displaying of the dissipation area.

Figure 7.5 shows one-dimensional energy spectra obtained during experiments in various laboratories, and flow has of various types (boundary layers, pipe channel, grid) and is characterized by various values \( \text{Re}_\lambda \) (last value in the legend). Experimental data are highlighted by point marks, and the related lines illustrate the model spectra for the values \( \text{Re}_\lambda = 30, 70, 130, 300, 600 \) and 1500. For displaying the Kolmogorov non-dimensional coordinates were used. The spectrum was normalized in the formulation \( E_{11}(\kappa_1)/(\epsilon \nu^{5/3})^{1/4} \) and the wave number \( \kappa_1 \eta \). We can see that these coordinates enable the fusion of spectra for \( \kappa_1 > \kappa_{EI} \), meaning in the inertial
subrange and in the area of dissipation. Individual cases differ in the extent of the inertial subrange, and generally their extent gets larger with a growing Reynolds number \( Re_\lambda \). The dissipation area \( \kappa \eta > 0.1 \) then merges for all examined cases.

\[
E_\kappa (\kappa \eta) = \lambda_\kappa (\kappa \eta)^{1/4}
\]

**Figure 7.5 – Energetic one-dimensional spectra of various turbulent flows**

Another important statistical characteristic is the dissipation spectrum, which illustrates the dissipation of energy in individual wave numbers. The dissipation spectrum is schematically illustrated in Figure 7.6 together with the energy spectrum for the specific value of the Reynolds number, and the vertical axis is linear here.

\[
\kappa \eta
\]

**Figure 7.6 – Energy and dissipation spectrum**
6.3.2. **Energy cascade and inverse cascade**

Let us look more closely now at the flow of energy in developed isotropic flow. From an energy point of view, the flow fluid represents an open system. Energy is carried into the system from the main stream, and through the mechanism of loss of stability, large vortex structures are created. Their specific form can be attributed to the boundary conditions of the particular case. These vortices characterized by the scale $l_0$ and the corresponding wave number $\kappa$ stand at the top of the energy cascade and represent the energy area of the wave numbers.

This is followed by the transfer of energy from large scales towards lower speeds $\varepsilon$ inside the inertial subrange. This is an energy cascade during which quick growth of the isotropy of the topology of vortex structures occurs. The mechanism for the transfer of energy is the stretching of vortices, as has already been stated. The process ends in the area of dissipation on the Kolmogorov scale $\eta$. A diagram of the entire process is shown in Figure 7.7. According to the original Kolmogorov theory, vortices at all levels of the cascade evenly fill the entire space.

![Figure 7.7 – Energy cascade](image)

However, situations can occur when the motion of fluid in one direction is disabled by boundary conditions, and then the motion must be two-dimensional in particular scales. This involves, for example, motion in thin layers of fluid, and the case of the Earth atmosphere also resembles this situation. Flow under these conditions may be spatial only for certain scales, which are smaller than the thickness of the layer. The structures of larger scales can occur only in directions without limiting effects, and their vorticity is oriented in the direction of the smallest dimension – the thickness of the layer. The stretching of these vortices and generation of vorticity cannot occur, and the value of vorticity in these scales is conserved.

For two-dimensional turbulence, a theory exists that was proposed in 1967 by Kraichnan. This theory is equivalent to the Kolmogorov theory for spatial turbulence. Energy is transferred on wave numbers $\kappa_i$ with rate $\varepsilon$, along with enstrophy of rate $\beta$, which is of order $\kappa_i^3 \varepsilon$. This enstrophy moves cascadingly in the direction of small scales of rate $\beta$ and eventually upon reaching the Kolmogorov scale it dissipates. Kinetic energy cannot be transferred in the same way as enstrophy, which affects it as a defense. Therefore a reverse cascade of energy is created from wave numbers $\kappa_i$ towards lower values. This process is characterized by rate $\varepsilon$ and the mechanism of pairing vortices is applied during it. This theory considers not only conservation of energy in the inertial range, but also the conservation of enstrophy, which applies only in the area of two-dimensional turbulence. In the same way that Kolmogorov uncovered the energy cascade, Kraichnan predicted the direct cascade of enstrophy from large scales to small scales characterized by the $\kappa^{-3}$ energy spectrum and an inverse energy cascade from small scales to
large, which gives $\kappa^{-5/3}$ an energy spectrum. However, the confirmation of this theory through experimental data and results of direct numerical simulation is not clear, and the inclination of the energy spectrum often turns out somewhat steeper than from the theory ($\kappa^{-3}$).

The spectra corresponding to the direct and reverse cascade are highlighted schematically in Figure 7.8, and energy flows in both cases are shown.

### 6.3.3. Internal intermittence

Kolmogorov theory K41 described above is based on very strict and as shown by many experiments also non-physical expectations, which have often been criticized. For example, Landau criticized Kolmogorov’s expectation of a constant value of the rate of transfer between scales $\varepsilon$ independent of the size of the scale and argued that the dissipation field should also be considered random. Also based on the results of experiments, in 1951 Townsend described the intermittence of small scales. Based on these arguments, Kolmogorov carried out a revision of his theory and in 1962 formulated a theory labeled as K62. This model works with the energy spectrum in the following form

$$E(\kappa) = C \varepsilon^{2/3} \kappa^{-5/3} \ln \left( \frac{\kappa}{\kappa_i} \right)^{\beta},$$  \hspace{1cm} (7.48)$$

where $\kappa_i$ is the wave number on which the energy is carried from the flow (the inverse value of the integral scale). Kolmogorov proposed a model for the transfer of energy as a multiplicative random action, during which only a portion of energy is carried that is characterized by the fraction $\beta$ from one scale to another. He also expected that the density of the probability of the dissipation field would change randomly in space and time according to the log-normal division of probability.
It is apparent from the obtained experimental results that Kolmogorov’s expectation of isotropy of small scales is in fact not correct. Evidence of this conclusion is the existence of intermittence in the area of small scales, which is a rule rather than an exception. The result is certain discrepancies during the evaluation of statistical moments and structure functions of higher order.

Despite its simplicity, the K41 theory brings a surprisingly good agreement with the experiment. However, as of the beginning of the 1960s it became apparent that its validity was only approximate and that refinement of experiments did not lead to better agreement with theories. An effort was made to make additional corrections to the theory. One of the most well-known problems discovered was that the scaling exponents of structure functions grew more slowly than predicted by Kolmogorov law (see 7.3.4). The corrected Kolmogorov theory is much more complex than theory K41, but the agreement with experimental data is not better with the new theory.

Theory K62 is based on the assumption that the level of dissipation is not constant, but that it depends on a scale, and its logarithm \( \ln(\varepsilon(r)) \) is a random quantity characterized by Gauss distribution. The emerging law of the resulting division of probability is called lognormal.

Other models gradually emerged (such as the beta model, the log-Poisson model) and continue to emerge, without this problem, called internal intermittence, having been definitively solved so that the predictions of the theory can be confirmed in experiments.

6.3.4. Formulation for structure functions

The purpose of the structure functions defined in Chapter 7.1.3 is to measure the intensity of fluctuations depending on a scale directly in the space of coordinates.

Kolmogorov theory predicts the existence of an inertial range, which is characterized by known law \(-5/3\) for spectral density of kinetic energy. It has been shown that a similar law also applies for structure functions.

The definition formula of the structure function (7.21) tests the exponential powers of growths in velocity between points spatially distant by \( r \) (modulus of vector \( r \)). In the inertial range, these structure functions must depend only on \( \varepsilon \) and on scale \( r \). From there stems scaling

\[
S_n(r) \sim \varepsilon^{n/3} r^{n/3},
\]

which with Kolmogorov’s expectation that \( \varepsilon \) does not depend on \( r \) is expressed in detail as

\[
S_n(r) \sim r^{n/3}.
\]

One decade of the spectrum apparently actually behaves according to the scale relationship \( E(\kappa_i) \sim i^{-5/3} \sim \kappa_i^{-5/3} \). This area of wave numbers is considered an inertial range. Practically achievable are only values \( r_i = i\bar{u}\Delta t \), where \( \Delta t \) is the inverse value of the sampling frequency of recording. If this frequency is not high enough, it will not reach small scales. It has been shown that the difference can usually be replaced with the absolute value of the difference and that the scaling does not change. Figure 7.9 shows several structure functions of the same signal depending on index \( i \), which is proportional to \( r \).
As is visible in Figure 7.9, unlike the energy spectrum, even in the structure functions of the lowest order, the area of scaling $r^\zeta$ is not clearly apparent. It has been shown that the inertial range is much less apparent in the structure functions than in the spectrum. For this reason, the hypothesis of extended self-similarity is used, which is based on the fact that the structure function $S_3(r)$ is scaled theoretically as $r$ and for obtaining the scaling exponents the relationship

$$S_n(r) \sim r^{\zeta_n},$$

is used, which is shown as almost fully fulfilled. According to K41, exponent $\zeta_n$ linearly depends on $n$ according to relationship $\zeta_n = n/3$, but in reality it is a non-linear function of $n$. However, for low orders of structural functions this deviation is very small, and it is more important only for orders 3 and greater. These deviations are related to the intermittence of small scales.

### 6.3.5. Turbulent diffusion

The diffusion of fluid particles characterizes the process of mutual distancing of two fluid particles and typically involves a Lagrange problem. The classic theory of diffusion offers two mechanisms – coherent and incoherent.

Incoherent diffusion is basically molecular diffusion, which is related to thermal agitation of molecules. For incoherent diffusion there is a simple one-dimensional model, a “drunken walk” model. Let us imagine that a drunken person at the beginning of his attempt is located at the start of a coordinate system and is capable of making an unlimited number of steps that are equal in length, and the probability that each subsequent step will be forward or backward is the same. After a large number of steps, the most probably position of the drunk is the same as at the beginning of his attempt. The probability of other positions is subject to normal, Gauss division of probability with a maximum at the beginning. This is how each gas particle behaves in a 3-dimensional space, and the resulting distribution of particles is in accordance with Gaussian 3D distribution. However, for examining diffusion, a more important piece of data is the variance of the distance of the particles, meaning the value $\langle r^2 \rangle$, and it for this spatial incoherent case is proportional to the time interval $t$.

On the contrary, coherent diffusion is characterized by the equal distance of two particles of constant velocity $d\langle r \rangle/dt \sim \text{const}$, and their distance is a linear function of time $\langle r \rangle \sim t$, to which the dependency $\langle r^2 \rangle \sim t^2$ corresponds.
It has been shown that for turbulence in the inertial regime, which is characterized by the spectrum \( E(\kappa) \sim \kappa^{-5/3} \), the average velocity of the distance of two particles is \( \frac{d\langle r^2 \rangle}{dt} \sim r^{1/3} \) and for the time change of the symmetrical deviation, the known law 4/3 applies:

\[
\frac{d\langle r^2 \rangle}{dt} \sim \left( \sqrt{\langle r^2 \rangle} \right)^{4/3}.
\] (7.51)

Therefore, in relation to time we arrive at the relationship \( \langle r^2 \rangle \sim t^{1/3} \). Turbulent diffusion is therefore the most effective of all specified situations.

These relationships were first published by Richardson in 1926 based on observation of the behavior of weather balloons in the atmosphere. Richardson diffusion is often called abnormal (or anomalous) and differs from both of the classic methods of diffusion described above, meaning from coherent and incoherent.

Richardson’s result was confirmed by Kolmogorov in his theory K41. At present more modern methods of modeling turbulent diffusions are used, which are based on the theory of Markov processes, but the basic idea is the same.

### 6.4. Dynamical systems

The flow of fluid is described by the system of partial differential equations (N-S equations). It is known that partial differential equations can be approximated by a system of ordinary differential equations (such as with the help of the Galerkin method). If we want to describe the system sufficiently accurately, then we should consider the high number of these equations as performed during direct numerical simulation. For example, for completely developed turbulence, the necessary number of equations is proportional to the Reynolds number in a certain power (e.g. \( \text{Re}^{9/4} \)). However, if attention is focused only on the rule of scaling and the basic mechanisms of the cascade transfer of energy in the area of small scales, then it is possible to capture the basic phenomena with the help of a limited number of ordinary differential equations. This involves shell models, and the basic idea for them originated from Obuchov (1971). N-S equations are replaced by a dynamical system with \( N \) variables \( u_1, u_2, \ldots, u_N \) of which each represents typical amplitude of the velocity field belonging to a certain length scale. Fourier space is divided into \( N \) shells. Each shell \( \kappa_n \) is comprised of a set of wave vectors \( \kappa \) that are such that \( \kappa_0 2^n < |\kappa| < \kappa_0 2^{n+1} \). Variable \( u_n \) is the difference in velocity at length \( \sim \kappa_n^{-1} \), and so for each shell we have one degree of freedom. However, modifications exist with more degrees of freedom for each shell (Grossmann a Lose, 1992). An effort was made to find a special scheme for closing the system of equations, which would be capable of reproducing the Kolmogorov spectrum in the sense of a certain point of a suitably differential equation for the field velocity averaged within the shell in Fourier space. Variable \( u_n \) is determined by mean energy belonging to the \( n \)th wave number

\[
u_n(t) = \left( \int_{\kappa_n}^{\kappa_n^+} 2E(q,t) dq \right)^{1/2},
\] (7.52)

can be considered an increase in velocity \( |u(x) - u(x + \ell)| \) for vortices of size \( \ell \sim \kappa_n^{-1} \). Then the shell by dimension

\[
\kappa_n = r^n \kappa_0 \text{ during } r > 1
\] (7.53)

contains wave numbers with modulus \( \kappa \) which are such that \( \kappa_n < \kappa < \kappa_{n+1} \) and standardly \( r = 2 \) is selected. Then the wave numbers \( \kappa_n \) are equidistant on the logarithmic scale.

During the construction of a corresponding equation, the following conditions must be fulfilled:

- The linear term for \( u_n \) is \(-\nu \kappa_n^2 u_n \);
- The non-linear terms for \( u_n \) are based on the quadratic form of type \( \kappa_n u_{n'} u_{n''} \);
• Waking and muffling terms are not present, and therefore there must be conservation of energy $\frac{1}{2} \sum_n |u_n|^2$;

• Interactions between shells are local in the $\kappa$-space, meaning $n'$ and $n''$ are close to $n$.

These conditions stem directly from the N-S equations, and the last is one of the possibilities for closing the system.

Desnyansky and Novikov (1974) introduced a model in which energy is conserved and interactions occur only between neighboring shells. These conditions lead to the following equation for real variable $u_n$:

$$\left( \frac{d}{dt} + \nu \kappa_n^2 \right) u_n = \kappa_n \left[ u_{n-1}^2 - 2 u_n u_{n+1} - 2^{3/5} \left( u_{n-1} - u_n - 2 u_{n+1}^2 \right) \right] + f_n,$$

(7.54)

where $n = 0, 1, ..., \infty$, boundary condition: $u_{-1} = 0$ and the external waking in the first shell does not depend on time: $f_n = f \delta_{n,0}$, and parameter $C$ is not constant and is based on the type of asymptotic scaling of variable $u_n$.

Independent of value $C$ we reach for the limit case, when the number of shells goes beyond all limits, the following spectrum

$$E(\kappa) \sim \kappa_n^{5/3} F(\kappa_n / l_0),$$

(7.55)

where $\kappa_D = (\epsilon / \nu^3)^{1/4}$ is the inverse value of the Kolmogorov scale and dissipation of energy is

$$\epsilon = \sum_n f_n u_n = f u_0.$$  

(7.56)

Then for $C < 1$ we reach for $\lim_{x \to 0} F(x) = F_0 \neq 0$ the known Kolmogorov 5/3 law. Whereas for $C > 1$ the function $F(x)$ is not analytical in $x$ for $x \to 0$, then we reach $F(x) \sim x^{\tilde{\xi}}$, where $\tilde{\xi} = 2 \log_2 C$. In the limit for $\nu \to 0$ the corresponding spectrum is

$$E(\kappa) \sim \kappa^{-(5/3 + \tilde{\xi})}$$

(7.57)

and the rate of dissipation is then

$$\epsilon \sim V^{9/2(8 + 3\tilde{\xi})}.$$  

(7.58)

We can see that the rate of dissipation is reduced to zero with the reducing viscosity, and energy is not dissipated on small scales, but can be cascadingly carried to greater scales. This conflicts with the observation in which it is shown that for $\text{Re} \to \infty$ the rate of dissipation settles at a non-zero value. This simple model also does not capture the effect of intermittence.

Another popular model is the “GOY” model (named after the authors Gledzer, Ohkitani and Yamada), which is already capable of modeling intermittence. This model works with complex variables.

The spatialness of vortex structures makes it possible to capture “hierarchic models”, which introduce more variables for a single shell. Very complex models have already been calculated, which are not entered very much with direct numerical simulations with the help of pseudospectral methods. However, they have the advantage that with their help it is possible to model flow to significantly higher values of Reynolds numbers than is real by using a direct numerical simulation.

Shell models basically represent shortened N-S equations, but the main difference compared to full equations is the fact that the wave number $\kappa$ here is scalar, and therefore information is lost about the spatialness of vortex structures. We will later examine the dynamical system with chaotic behavior, which is characterized by a reasonable number of degrees of freedom (typically lower than 100). An advantage is the possibility to study the corresponding strange attractor in the phase space with the help of standard analytical tools used during the analysis of deterministic chaos, such as fractal dimensions, the Lyapunov spectrum, dynamic intermittence and multifractality. This explains the connection between the ergodic properties of the dynamics of strange attractors and the actual three-dimensional space with an infinite number of scales.
7. Examples of turbulent flows

In this chapter we will list typical examples of turbulent flows. We will also focus on the properties that relate to their turbulent structure.

7.1. Grid turbulence

Let us look first at grid turbulence, which is a somewhat atypical example of turbulent flow, because it does not involve a shear area in the sense of a field of mean velocities. Grid turbulence occurs behind a planar grid, which is manufactured from cylindrical rods, which form square mesh. The size of an mesh of the grid is $M$, and the diameter of the rod is $d$.

If fluid flows through the mesh of the grid perpendicularly to its level of mean velocity $U_0$, gradually (at distance $x_1$ approximately $15 \cdot M$) a homogeneous isotropic turbulent flow occurs (see Figure 3.10). Behind individual parts of the grid, wakes appear, which are focused in two mutually perpendicular directions. They then interact with each other, so that relatively gradually homogeneous and isotropic turbulence is created. Homogeneity is fulfilled here in directions $x_2$ and $x_3$, isotropy in all 3 directions is related to fluctuation parts of velocity and mean velocity is at a non-zero value only in direction $x_1$.

![Isotropic fluctuations](Figure 8.1)

In the case of an isotropic field of fluctuations in velocity, in coordinates $(u'_1,u'_2,u'_3)$ we move inside a sphere; a two-dimensional analogy is shown in Figure 8.1. Then it can be expected that the parts of the fluctuation velocity will be non-correlated, meaning $u'_i u'_j = 0$ for $j \neq i$ and it also applies that $u'^2_1 = u'^2_2 = u'^2_3$, and then the tensor of Reynolds tension is formulated $-\rho u'^2_i$. This is a case of grid turbulence when production of Reynolds tension does not occur. The energy balance here has a simplified formulation, because in the flow field as a result of the validity of the continuity equation mean velocity is constant everywhere, and therefore production is zero. Therefore, there must be suppression of the kinetic energy of turbulence in the direction of the mean stream $x_1$. Experiments show the power dependency in the form

$$\frac{k}{U_0^2} = K \left( \frac{x_1 - x_{10}}{M} \right)^n,$$

where $k$ is the kinetic energy of turbulence, $U_0$ is the mean velocity of flow, $K$ is a parameter whose value depends on the geometry of the grid and on the Reynolds number, $x_{10}$ is the position of the grid, $M$ is an mesh of the grid, and $n$ is an exponent, whose value is from experiments in the range of $1.15 < n < 1.45$.

Based on the law of suppression of free isotropic turbulence, it is possible to define the dissipative scale of $Le$ turbulence
This scale is an important characteristic of turbulent flow and plays an important role in the processes that influence free turbulence, such as the shortened transfer of the boundary layer to turbulence caused by turbulence in the external running flow.

7.2. Shear flows

In Chapter 6, it was shown that for the creation of an unstable condition in fluid, a non-homogeneous structure of flow fluid is necessary, which is typical for shear areas, when there are changes in the velocity of flow across the flow field. If these changes did not occur, then it would be apparent from the continuity equation for incompressible fluid that the velocity of the fluid must be constant in the whole area. Therefore, let us now discuss some typical canonic examples of turbulent shear flows.

During flow near solid surfaces, a shear area is created as a result of the phenomenon of fluid being spit on the wall, when on the wall the vector of velocity of the flow of the fluid is zero. A basic type of a wall shear area is a boundary layer. A boundary layer occurs close to a wall surrounded by flow fluid.

The profile of mean velocity in a turbulent boundary layer has a typical shape, which is shown in Figure 8.2; here the non-dimensional coordinates $u_*$ and $y_*$ are introduced with the help of friction velocity. The definition of friction velocity $u_*$ and non-dimensional coordinates of the boundary layer is as follows

$$u_* = \frac{\tau_0}{\rho} , \quad u_* = \frac{\bar{u}}{u_*} , \quad y_* = \frac{x_* u_*}{\nu} ,$$

(8.3)

where $\tau_0$ is the time-related mean value of shear friction on the wall, $\rho$ is the density and $\nu$ is the kinematic viscosity of the fluid. The horizontal axis in the graph in Figure 8.2 is logarithmic.

Near the wall a viscous sublayer is created, where there is linear dependency between the parameters of the boundary layer. Additionally from the wall here is a logarithmic region, and
between these two areas there is a buffer layer. The buffer is defined by the values of non-dimensional distance from the wall $5 < y^* < 30$, and the total non-dimensional thickness of the turbulent boundary layer can be of the order $10^3$.

We shall assume that the statistical quantities have in the turbulent flow a planar distribution that does not depend on the value of coordinate $x_i$, and the fields are symmetrical according to this axis. For distribution of the probability of velocity $f(u, x, t)$ the following applies

$$\frac{\partial f}{\partial x_i} = 0,$$  \hspace{1cm} (8.4) $$f \left( u_1, u_2, u_3; x_1, x_2, x_3; t \right) = f \left( u_1, u_2, -u_3; x_1, x_2, -x_3; t \right).$$  \hspace{1cm} (8.5)  

These two equations present for $x_i = 0$ the evenness of the mean velocity $\overline{u_3} = -\overline{u_2}$, and this velocity can be fulfilled only for $\overline{u_3} = 0$. Relationships can be similarly arrived at for parts of Reynolds tension $\overline{u_i'u'_i} = 0$ and $\overline{u_j'u'_j} = 0$. It is apparent from equation (8.4) that these relationships must apply in the entire area.

For a nominally planar flow, we get a zero level of mean velocity in the direction $x_i$ and the tensor of Reynolds tension will be formulated as

$$\begin{bmatrix}
  u_{i1}'^2 & u_{i1}'u_{i2}' & 0 \\
  u_{i1}'u_{i2}' & u_{i2}'^2 & 0 \\
  0 & 0 & u_{i3}'^2
\end{bmatrix}.$$  \hspace{1cm} (8.6)  

The correlation of the part of fluctuation velocity $u'_i$ with the remaining parts is zero.

Now let us look more closely at the planar boundary layer on the board. The main flow is directed here along the board in the direction of axis $x_i$. For stationary flow, we can expect the following to apply:

$$\overline{u_1} \gg u_2 \text{ a } \frac{\partial}{\partial x_1} \ll \frac{\partial}{\partial x_2}.$$  \hspace{1cm} (8.7)  

The Reynolds equation is reduced to a single equation for the direction $x_i$

$$\frac{\partial \overline{u_1} \overline{u_1}}{\partial x_1} + \rho \overline{u_1} \overline{u_{i2}} = - \frac{1}{\rho} \frac{\partial P}{\partial x_1} + \frac{1}{\rho} \frac{\partial}{\partial x_2} \left[ \mu \frac{\partial \overline{u_1}}{\partial x_2} - \rho u_{i1}'u_{i2}' \right].$$  \hspace{1cm} (8.8)  

The gradient of pressure in the direction of the main flow is often zero. The expression in the bracket on the right side represents the total shear stress tension $\tau_{tot}$, which is comprised of the viscous part of $\tau$, and the turbulent part of $\tau$:

$$\tau_{tot} = \tau_v + \tau_t = \mu \frac{\partial \overline{u_1}}{\partial x_2} - \rho u_{i1}'u_{i2}'.$$  \hspace{1cm} (8.9)  

Near the wall, specifically in the viscous sublayer, the buffer and the logarithmic region, the total shear tension has an approximately constant value, which is equal to the size of the shear friction on the wall $\tau_w$, and further from the wall its value almost linearly declines. On the wall the turbulent part of the shear friction $\tau_t$ is zero (because the fluctuations of velocity are zero), and the viscous part $\tau_v$ here is equal to the surface friction. If we move now from the wall, the viscous part declines and the turbulent part grows, until in the logarithmic region the viscous part has an insignificant size compared to the turbulent one. The situation in the turbulent boundary layer is illustrated in Figure 8.3. Axis $x_i$ lies on the wall, the viscous part of the shear tension $\tau_v$ is highlighted in gray. It is apparent from the Figure that the viscosity of the fluid influences the turbulent boundary layer only in direct proximity to the wall in the viscous sublayer. This effect resounds in the buffer and nearly vanishes in the logarithmic region.
In the boundary layer, the turbulent part of shear tension $-\rho \overline{u'u''}$ and the gradient of mean velocity $\partial \overline{u}/\partial x_2$ in the transverse direction almost always has the opposite sign. The situation is illustrated in Figure 8.4. Fluid particle $A$ moves downward from the position $x_{2A}$ to the position $x_{2B}$, forming fluctuation of velocity $u'_2<0$. This also creates in location $B$ fluctuation of velocity $u'_2>0$, because $\overline{u_{2A}}>\overline{u_{2B}}$ applies. Therefore the correlation of $u'_1$ and $u'_2$ must be negative, and the value corresponding to the Reynolds tension is positive. For the fluid particle in location $B$, which moves to $A$ the situation is exactly opposite: the fluctuation of velocity $u'_2>0$ and fluctuation of part $u'_1<0$. The result is therefore the same – the correlation of $u'_1$ and $u'_2$ must be negative, and the value corresponding to the Reynolds tension is positive. This leads to the generation of Reynolds tension. The situation is illustrated in Figure 8.5. The situation is similar in the free shear layer.
There is a similar situation in the case of a planar flow in a channel, in a pipe or in a free stream or wake, but there of course a viscous sublayer is not created.

Now we will focus on the energy balance in Reynolds’ description, which was formulated in detail in Chapter 5.23. Generally the equation of the balance of turbulent energy is expressed in the stationary case of equilibrium of 4 terms

$$A + P + T + D = 0,$$

(8.10)

Where $A$ is advection, carrying via mean flow (this term has the general formulation $-u_i \frac{\partial}{\partial x_i} \left( \frac{1}{2} q^2 \right)$), $P$ is production of turbulent energy related to mean flow, is usually in the form

$$-u_i^i \frac{\partial u_i}{\partial x_j},$$

$T$ is turbulent transport, which relates to fluctuations of pressure

$$\frac{\partial}{\partial x_j} \left[ u_i^i \left( \frac{1}{2} q^2 + \frac{p'}{\rho} \right) \right]$$

and $D$ is dissipation of turbulent energy related to viscous effects and is characterized by the rate of dissipation $\varepsilon$. Details regarding the balance of energy were presented in Chapter 5.

In relation to a turbulent boundary layer, this balance is qualitatively illustrated in Figure 8.6. The production is the greatest in the buffer, like dissipation, and both are reduced in the direction of the border of the boundary layer. Transport has an effect in the boundary layer of loss, and advection inside the boundary layer causes losses, but gains at its edges.

Free shear areas are created during the interaction of two or more flows in a free space and directly interact with the restricting walls. The basic types of free shear areas are jet, mixing layers and wakes. A jet is basically a combination of mixing layers around the core.
A jet is created as a result of the interaction of flow fluid, which flows out of an opening (nozzle) with flow fluid, which can be in a calm state. At the borders of the jet, new shear layers are created, which are usually unstable, and they quickly expand in a transverse direction until they merge. Then a typical velocity profile of mean velocity of a bell shape is created. It has been shown that further along the flow, this profile as well as profiles of turbulent characteristics have the property of self-similarity. This means that the form of profiles of these quantities does not depend on coordinate $x_1$, but it is necessary to perform transformation of these profiles. In suitable non-dimensional coordinates, we reach a universal form of these profiles.

![Figure 8.7 – Balance of energy in turbulent jet](image)

Let us look again at the energy balance across the flow. Figure 8.7 shows the development of individual terms of the energy equation for the planar turbulent jet in the transverse direction $x_2$. The case is symmetrical for negative values of the coordinate.

A wake, or area of recirculation flow, occurs in connection with tearing away of the boundary layer from the wall. The form of the energy balance in the case of a wake is very similar to the case of a jet shown in Figure 8.8.

![Figure 8.8 – Balance of energy in turbulent wake](image)

The production term in the case of a jet and a wake has a maximum for a certain value of $x_2$. It can be shown that the position of this maximum is close to the maximum shear for the profile of mean velocity $\left|\frac{\partial \bar{u}}{\partial x_2}\right|$. This of course is not a coincidence, and two interpretations of this phenomenon are possible. First, the production term itself contains this derivative, and it can be
assumed that its maximum is close to the maximum of this derivative. Also, close to the maximum derivative, there is the position of the inflection point on the profile of velocity $\bar{u}$, and according to the inviscid theory of stability, the flow is unstable, and turbulent energy is produced via this mechanism.
8. Modeling of turbulence

Methods of mathematical modeling usually use methods of end volumes and suitable numerical schemes; this problem is not contained in this discussion, and the reader can find details in corresponding literature.

The implementation of all methods used in turbulence is based on the use of methods of numerical mathematics – discretization of the problem is performed in space and time. Generally each method of mathematical modeling requires enough spatial and time discretization to be able to model the values of the gradients of all variables, which are taken into consideration in this case. The maximum values of these gradients are based on the minimum sizes of structures. Their sizes depend on conditions of flow (the geometry of the area, speed, properties of the fluid) and on the requirement for the results. For example, with the use of the DNS method, it is necessary to select discretization so that vortex structures of all sizes up to Kolmogorov length and Kolmogorov time scale are captured. In the LES method of discretization, the filter for spatial structures is basically defined. The least demanding for the number of discretized elements are the RANS methods, because spatial gradients of mean variables have much lower values in turbulent flow than instantaneous gradients.

Discretization of the problem in space and time is closely connected. Time discretization can be such that the model is capable of capturing the dynamic behavior of structures, which are modeled on a spatial net. This means that the smaller the element of discretization of the network is, the smaller the structures are that we simulate and the smaller the time step is that we have to use. This is directly related to the speed of the changes of these smallest structures.

For numerical resolution of differential equations, we use regular computers, which, however, are of the von Neumann type and work on the sequence principle. It can generally be stated that N-S equations are elliptic and contain both spatial and time connections. The task for their solution is suitable for massively parallel algorithms and very unsuitable for sequential algorithms. For their resolution, which is currently extremely demanding on the performance and speed of sequential computers, there are great prospects for the development of parallel algorithms and mainly parallel computers.

![Graph](image.png)

*Figure 9.1 – Result of solution of N-S equations*

We currently have three methods available for solving N-S equations. They are direct simulation of N-S equations (Direct Numerical Simulation – DNS), when we solve a problem in space and time, the Reynolds formulation for solving mean fields (Reynolds Averaged Navier-Stokes – RANS) and a combination of both approaches, when we simulate large vortices and model small structures with the help of Reynolds equations (Large Eddy Simulation – LES). The result obtained with the help of individual approaches in the particular point in space is illustrated only on Figure 9.1. The DNS method provides the exact time course of the monitored variable,
and the RANS method provides only the mean value in time, and the result of the LES method is a smoothed signal.

In practice, the border between the methods described above is not sharp, and different variance exist, such as “Very Large Eddy Simulation” (VLES) and “Unsteady RANS” (URANS), which represent transitional stage between the RANS and LES methods.

The selection of the method of mathematical solution for the specific task is usually predetermined by practical points of view. DNS are taken into consideration only if the task is characterized by simple geometry, the Reynolds number is very low and we also have necessary hardware available. When solving the overwhelming majority of practical engineering tasks, we can be satisfied with the RANS approach, and only in exceptional cases can the LES method be applied; the DNS method is reserved for special tasks of a research character.

Despite the entirely fundamental theoretical problems related to RANS methods, while maintaining a proper approach it is possible to achieve very good results that are useable in practice with the help of these methods. However, these calculations should always be complemented with verification experiments performed best for the same task, or at least a qualitative comparison with published results of experiments during similar tasks. For an entirely unknown method of resolving these tasks, these methods are generally not suitable. In relation to this, it needs to be emphasized that methods such as RANS generally cannot be used for modeling instability of any time, due to the inability of these methods to model the development of small perturbations in time and space. RANS methods cannot be used to reliably predict either the transition to turbulence or the tearing away of the boundary layer. These phenomena need to be modeled with other specialized models, or data about them needs to be provided from experiments. However, in areas where the conditions are fulfilled for proper functioning of the selected RANS model, these methods work very reliably.

### 8.1. Direct Numerical Simulation (DNS)

Direct numerical simulation represents modeling of N-S equations with the help of numeric mathematical processes. Spatial and time discretization needs to be selected to ensure covering of the entire spectrum of vortex structures that occur in real flow. This means that for the discretization network we must be able to capture vortices using the dimension of the Kolmogorov scale.

In relation to this, the number of degrees of freedom \( n \) of the particular problem is usually defined, which is related to the number of elements of the discretization network. It is based on the relationship between the size of the largest \( \ell_0 \) and the smallest \( \eta \) structures in the flow field (see 7.3). From the Kolmogorov theory for isotropic turbulence, there is apparent dependence of this relationship on the Reynolds number in the formulation \( \ell_0/\eta \sim \text{Re}^{3/4} \). The number of degrees of freedom of the problem \( n \) is related to the number of elements in the 3-dimensional space. Therefore the following proportion applies:

\[
 n \sim \left( \frac{\ell_0}{\eta} \right)^3 \sim \text{Re}^{9/4}.
\]  

(9.1)

It is obvious that it does not make sense to perform simulation in a reduced number of dimensions (at a level), but always in a 3-dimensional space, because turbulent flow is always of a spatial character.

During the implementation of calculations with the help of DNS, spectral or pseudospectral methods are used. A solution is expected in the form of a Fourier series in space. This approach was processed in 1972 by Orszag with Patterson, and as a result of the capacity of computer technology, the DNS method is still used for calculation of geometrically simple areas of flow during extremely small Reynolds numbers. For increasing the Reynolds number, the number of discretized elements in the area is sharply growing, and the necessary time step is declining. The calculations of flow with the help of DNS are extremely demanding on the performance of computer technology and last a very long time.

It can be proved that computer demands with consideration for the discretization in space and time grow with the sixth power of the Reynolds number. Today’s best computers with performance of dozens of gigaflips are capable of resolving tasks with the help of DNS...
characterized by a Reynolds number of order of a maximum of $10^3$, and interesting from an application point of view tasks of turbulent flow in machines are characterized by Reynolds numbers of the order $10^5$ and greater, and flow in the atmosphere and hydrosphere is several times greater. For achieving a ten times greater Reynolds number, it will be necessary to increase the performance of computers a million times.

The DNS method is currently used and obviously for a long time in the future will be used to solve fundamental tasks related to the theory of turbulence. It is limited to geometrically simple areas and very low Reynolds numbers. However, it provides us with a perfect picture of the physics of flow fluid. From a correctly performed DNS simulation, it is possible to obtain random variables in a random location and time. The results of such simulations are generally regarded as equivalent to results of experiments, but in terms of reliability and comprehensiveness of information, experiments are far ahead.

8.2. Large Eddy Simulation (LES)

The Large Eddy Simulation (LES) is based on the idea of filtering N-S equations. The problem is divided into two parts, and structures greater than a certain boundary size are modeled separately from smaller structures. However, both problems are mutually connected, and they cannot be solved separately.

We can generally state that large structures (vortices) are simulated directly like with DNS, and small structures are modeled similarly to the way they are using the RANS method. However, the model of small turbulent structures is somewhat modified in view of the limited size of modeled structures, and is a “subgrid model” – modeling structures smaller than the dimension of the network cell.

8.3. Methods of modeling Reynolds equations (RANS)

These methods are based on Reynolds Average Navier-Stokes (RANS) equations. As has already been stated, when deriving Reynolds equations, their problem is the inability to close the system of equations, when the number of equations is less than the number of unknowns, which compared to N-S equations was increased by Reynolds tension. Therefore, the modeling methods focus on closing the system of equations.

Methods of modeling turbulent flow with the help of RANS can be divided into two subgroups: methods using the idea of turbulent viscosity and the method of modeling Reynolds tension. Calculations can be successfully performed even in a shrunk dimension – for example for planar or rotational-symmetric flows.

8.3.1. Models based on turbulent viscosity

These models are very simplified. They capture only the effect of isotropic turbulent viscosity. They generally do not work well for flow with a non-isotropic turbulent structure. Surprisingly, boundary layers, which are not isotropic, fare relatively well, but they fail during modeling of quickened or slowed flows or as a result of curving of the wall. The mistakes mainly show up in the values of normal tension (intensity of turbulence). They are numerically stable (second category gradients in mean flow equations experience stabilization). They work with reasonable accuracy for engineering tasks.

Shown below are Reynolds equations, which we obtain through application of the operation of averaging for the N-S equation.

\[
\frac{Du_k}{Dt} = \nu \frac{\partial}{\partial x_k} - \frac{\partial \overline{u_i u'_k}}{\partial x_i} - \frac{1}{\rho} \frac{\partial \overline{p}}{\partial x_k}, \tag{9.2}
\]

\[
\frac{Du_i}{Dt} = \frac{1}{\rho} \frac{\partial}{\partial x_k} \left[ -\rho \delta_{ik} + \mu \left( \frac{\partial \overline{u_i}}{\partial x_k} + \frac{\partial \overline{u_k}}{\partial x_i} \right) - \rho \overline{u'_i u'_k} \right]. \tag{9.3}
\]

One of the classic methods of modeling Reynolds stress is a turbulent viscosity hypothesis which was introduced in 1877 by Boussinesq. This method uses an analogy with Newton law for expressing tension in fluid.
According to this hypothesis, tension in fluid, to which deviation anisotropic parts of Reynolds stress belong, is proportional to mean rate of the deformation of fluid particles in a way comparative to the situation with viscous forces. The constant of proportionality is “turbulent viscosity” \( v_t \).

\[-\rho u_i' u_j' + \frac{2}{3} \rho k \delta_{ij} = \rho v_t \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{9.4}\]

Reynolds equations then transfer to the formulation

\[\frac{Du_i}{Dt} = \frac{\partial}{\partial x_k} \left[ v_{\text{eff}} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \right] - \frac{1}{\rho} \frac{\partial}{\partial x_i} \left( \bar{p} + \frac{2}{3} \rho k \right), \tag{9.5}\]

where \( v_{\text{eff}}(x,t) = v + v_t(x,t) \) is the coefficient of effective viscosity. Let us notice that these equations formally have the same formulation as the N-S equations in which time mean velocity figures; the coefficient of molecular viscosity is replaced by an effective coefficient, and pressure is then modified by mean pressure \( \bar{p} + 2/3 \rho k \).

A problem that remains is the determination of turbulent viscosity, which is generally a function of position and time. During practical application, it is assumed that changes in turbulent viscosity in time can be overlooked, and for values of turbulent viscosity in space, recommendations exist that are valid for a certain category of tasks. Molecular viscosity is insignificant compared to turbulent viscosity.

The concept of turbulent viscosity assumes by default the even nature of normal parts of Reynolds stress and their isotropy. This condition is well fulfilled with developed turbulent flow, which is well isotropic, and in cases when shear parts of Reynolds stress dominate and normal parts are insignificant. In the case of a planar structure of a fluctuating flow field (hydrodynamic instability) this condition obviously is not fulfilled.

### 8.3.1.1. Algebraic models

According to the theory of mixing length, which was first formulated by Prandtl, the coefficient of turbulent viscosity is based on a multiple of the characteristic length of turbulence, which is often referred to as a “mixing length”, and the velocity characterizing the turbulent motions. Prandtl also assumed that the mixing length was proportional to the thickness of the turbulent area \( l \) and that the speed of the turbulent mixing motions \( \mathcal{U} \) was proportional to a typical value of the difference of mean velocities in the area:

\[v_t - \mathcal{U} l \approx l^2 \frac{\partial u_i}{\partial x_2}. \tag{9.6}\]

The mixing length is determined from a simple algebraic relationship, and therefore these models are called “algebraic” or also “zero-equation”.

Procedures based on algebraic models give good results in simple shear flows, such as boundary layers or simple wakes. For complex flows, these methods are generally unusable. The mathematical model exactly describes the behavior of the fluid, but the problem is determining the coefficient of turbulent viscosity, when for each point of the flow field it can be significantly different. Solving this problem is equivalent to solving the turbulent speed field, for which we have to use another method. Nonetheless, for a certain category of problems, these models are used even today, for example in surrounding of aviation profiles, where algebraic models of Baldwin-Lomax, Cebeci-Smith and Wilcox are used.

### 8.3.1.2. Models containing transport equations

More complex models work with transport equations for various turbulent variables. However, even in these models, the concept of turbulent viscosity based on Boussinesq assumption is used.

Transport equations for turbulent variables are derived from N-S equations. For example, for kinetic energy we reach a transport equation in the formulation
\[ \frac{\partial }{\partial x_i} \left( \rho u_i k \right) = -\rho u_i u_i \frac{\partial u_i}{\partial x_i} - \frac{\partial }{\partial x_i} \left[ u_i' p + \frac{1}{2} \rho u_i u_i' - \mu \frac{\partial k}{\partial x_i} \right] - \frac{\partial }{\partial x_i} \left( \mu \frac{\partial u_i'}{\partial x_i} \right). \] (9.7)

Term I is convection. Term II represents production, and large-scale vortices gain energy from the main flow. This term is often modeled with the help of equation (9.4). The first 2 readings in Term III represent turbulent diffusion occurring as a result of fluctuations of pressure and velocity, and the last represents viscous diffusion. The first pressure reading is usually very small, and therefore it is insignificant. The mean reading with a triple correlation is often modeled with the assumption that diffusion of kinetic energy is occurring against the direction of the gradient, meaning from locations with high kinetic energy to locations with lower energy. Therefore, we get

\[ \frac{1}{2} \rho u_i u_i' = -\frac{\mu}{\Pr_k} \frac{\partial k}{\partial x_i}, \] (9.8)

where \( \Pr_k \) is the turbulent Prandtl number for \( k \).

Term IV represents the dissipation of kinetic energy, which changes into heat. The estimate of the dissipation term tends to be expressed as

\[ \varepsilon = \frac{U^3}{\ell}, \] (9.9)

where \( U \) is the typical velocity of turbulent flows, and \( \ell \) is the dimension of vortices. We estimate the value of velocity from kinetic energy

\[ U = \sqrt{\varepsilon}, \] (9.10)

and the dissipation term IV from equation (9.7) will then be

\[ -\frac{\partial u_i'}{\partial x_i} = -\frac{k^{\varepsilon/3} \ell^2}{\varepsilon}. \] (9.11)

The transport equation for the rate of dissipation or for enstrophy tends to be modeled similarly. Derivations and the resulting form can be found in literature.

The simplest models using transport equations are “single-equation models”. In these models, the transport equation is resolved for a certain turbulent variable (usually kinetic energy), and another turbulent variable (usually the scale of turbulence) is gained from an algebraic relationship.

More complex models are two-equation models. In them, two transport equations are resolved for two scalars characterizing turbulent flows. These scalars can include kinetic energy, enstrophy and the rate of dissipation. The tensor of Reynolds stress is usually counted with the use of the Boussinesq hypothesis from gradients of velocity and turbulent viscosity. It is evaluated from the values of two scalars obtained from transport equations. One of the most well-known and in engineering work liberally used models is the \( k - \varepsilon \) model.

Flow near the wall can be modeled from a smoothening network in a manner that enables correct modeling of large gradients in this area. This approach is very demanding on computer capacity. Another option is the use of “wall functions”. We then predict that flow near the wall can be characterized as an entirely developed boundary layer. Wall functions are used in combination with two-equation models.

### 8.3.2. Modeling of Reynolds stress

The most complex models are used to model Reynolds stress directly (Reynolds Stress Models – RSM). Greater calculation demands are also apparent from their complexity. In these models, the transport equation is resolved for the tensor of Reynolds stress \( \overline{u_i' u_i'} \), and other transport equations tend to be used for the scale of turbulence and for the rate of dissipation \( \varepsilon \). These models are not based on Boussinesq hypothesis, and the equations for Reynolds stress are derived from N-S equations (see 5.5.1.).
9. Phenomenology of turbulence

In this chapter we will look more closely at the structure of turbulent flow. The basic element for a turbulent flow field is a vortex.

9.1. Kinematics

Let us examine the behavior of two fluid particles located close to each other in the flow field. Their velocities will be for the first point \( u_i(x_i, x_2, x_3, t) \) and for the second \( u_i(x_i', x_2', x_3', t) \). If we use Taylor development and drop terms of higher orders, then we can write

\[
u_i(x_i', t) = u_i(x_i, t) + \frac{\partial u_i}{\partial x_j}(x_i' - x_i), \quad (10.1)
\]

If we perform a decomposition of the second order tensor \( \partial u_i/\partial x_j \) to a symmetrical part and an asymmetrical part, then we get

\[
u_i(x_i', t) = u_i(x_i, t) + \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)(x_i' - x_i) + \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)(x_i' - x_i). \quad (10.2)
\]

Let us now introduce the usual label for the strain rate tensor \( S_{ij} \) and the tensor of rotation velocity \( \Omega_{ij} \)

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \quad (10.3)
\]

The antisymmetrical tensor \( \Omega_{ij} \) can be interpreted as pseudovector \( \omega_i \) of angular rotation

\[
-\omega_i = \frac{1}{2} \varepsilon_{ijk} \Omega_{jk} = \frac{1}{2} \varepsilon_{ijk} \frac{\partial u_j}{\partial x_k}. \quad (10.4)
\]

The vector of angular rotation \( \omega_i \) relates simply to the already defined vorticity \( \omega \):

\[
\omega = 2\omega_i = \nabla \times u \quad (10.5)
\]

If we state the definition (10.3) into equation 10.2, we reach an equation expressing the general motion of an elementary volume of fluid

\[
u_i(x_i', t) = u_i(x_i, t) + \varepsilon_{ijk} \Omega_{jk}(x_i' - x_i) + S_{ij}(x_i' - x_i). \quad (10.6)
\]

The first term on the right side of equation (10.6) represents the speed of translation of the elementary volume, the second term represents the speed of rotation, and the third represents the rate of deformation of fluid. The first two terms characterize the motion of the elementary volume of fluid, as if it involved a solid body.

The decomposition described above is a mathematical formulation of the first Helmholtz theorem, which says that we can split each motion of a fluid element in the surroundings of a certain point into translation (sliding) motion, rotation motion around a particular point and deformation motion.

Let us look now in greater detail at the properties of the field of vorticity \( \omega \). It is apparent already from the definition of vorticity that the field of vorticity in inviscid fluid is non-divergent, meaning

\[
\nabla \cdot \omega = \text{div} \omega = 0. \quad (10.7)
\]

We can characterize the field of vorticity using a set of curves comparable to flow in relation to the field of velocity. These curves are called “vortex lines”. They are defined as curves in fluid carried in a certain moment, when they have a property as a result of which in each of their points the vector of vorticity is their tangent. Mathematically, we can describe them using parametric equations (the parameter is time):
Let us assume now that only one vortex line passes through each point of the flow field. If within the flow fluid we have defined a closed curve, then one vortex line passes through each of its points (if the curve does not intersect with the flow line). If a curve is elected such that each of the vortex lines intersects it only once, then the vortex lines passing through the closed curve form a "vortex tube". The fluid contents of the vortex tube can be referred to as a "vortex filament".

Additionally, the flow of vorticity vector \( \omega \) through the cross section of the vortex tube can be referred to as "vortex tube intensity" or "vortex intensity". Let us now consider the cut of the vortex tube limited by two cuts \( S_1 \) a \( S_2 \) – see Figure 10.1. In view of (10.7) Based on the Gauss-Ostrogradsky theorem, it is possible to write:

\[
\iint_{S_1+S_2+S_3} \omega \cdot dS = \iiint_V \text{div} \omega dV = 0,
\]

where \( \omega \) is the part of vorticity perpendicular to the particular surface \( S \). The integral across the wall of the pipe \( S \) must be zero, because the value of the component of vorticity is zero in all points of this surface. It is apparent from this that the vorticity vector of cross sections \( S_1 \) and \( S_2 \) is the same.

This result is formulated by the second Helmhotlz theorem, which says that for a random cut by a particular vortex tube, in the particular moment its intensity is constant. It is directly apparent from this theorem that the vortex tube neither appears nor disappears in the fluid. This means that either it must reach the boundary of the flow area (the wall) or must be closed (ring). This applies exactly for inviscid fluid. The intensity of a vortex tube cannot be measured directly, and it is necessary to calculate it from the velocity field. For easier expression, we are introducing the term "velocity circulation". This is the calculation of the flow of the velocity vector along the closed curve bordering the cut of the vortex tube \( C \)

\[
\Gamma = \oint_C u \cdot ds,
\]

where \( s \) is an element of curve \( C \).

The value of circulation is identical to the value of intensity of the vortex tube, and we can convince ourselves of this easily using the application of the Green (Stokes) theorem

\[
\Gamma = \oint_C u \cdot ds = \int_S \text{rot} u \cdot n dS,
\]

where \( n \) is the unit normal vector of the elementary surface \( dS \).

For simple related areas, the following conclusions are arrived at from equation (10.11):

- If the flow of fluid in the entire considered area lacks vorticity, then the circulation of the speed along the random curve, which lies entirely in fluid, is zero.
- The circulation along the closed curve is not zero if it encloses at least one vortex filament.
- In flow that lacks vorticity, streams of the closed curve cannot be created.
If flow in any part of the area has vorticity then the circulation of velocity along random fluid closed fluid inside, meaning a closed curve, is equal to the sum of the intensities of vortex tubes, which intersect with the surface bordered by the curve. The penetrations of vortex pipes with this surface must cover the entire surface and must not overlap. The vortex tubes that intersect with the surface twice represent a zero contribution.

The vortex tubes (or filaments) move as material surfaces with fluid, meaning they are comprised of the same particles.

The following considerations apply exactly for a fluid that is incompressible, inviscid and barotropic (meaning such that its density is a function only of pressure). Let us now consider the time derivative of circulation along the "fluid curve". The fluid curve is a curve that during the motion of fluid is comprised of the same particles. We can express this derivative as follows:

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint u_i dx_i = \oint \frac{du_i}{dt} dx_i + \oint u_k \frac{d}{dt} (dx_k). \quad (10.12)$$

The second term on the right side of the equation (10.12) can obviously also be expressed as

$$\oint u_k \frac{d}{dt} (dx_k) = \oint u_k du_k = \oint \frac{1}{2} d (u_i u_k) = 0, \quad (10.13)$$

Therefore, the following resulting relationship applies

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint u_i dx_i = \oint \frac{du_i}{dt} dx_i \quad (10.14)$$

which is the mathematical formulation of Kelvin (Thomson) theorem: Time derivative of the circulation of velocity along a closed curve is equal to circulation of acceleration along the same curve. Kelvin theorem also applies for real viscous fluid. The result of Kelvin theorem is the validity of the following claim for the flow of incompressible, inviscid fluid in the field of conservative forces: if the motion of inviscid fluid at any time lacked vorticity, it will continue not to have it. This means that vorticity under these conditions is indestructible and impossible to create.

The theorems about vortices and the behavior of vorticity in an ideal inviscid fluid were formulated long ago. Even though they involve idealization, which during application gives inaccurate results regarding the behavior of fluid in a longer time segment, their significance is in the understanding of the behavior of vortex structures of a larger scale, where viscosity is not a dominant property, and in examination of the tendencies and motions of vortices in relatively short time segments.

9.2. Vortices

Vortices are a basic building block of the turbulent flow field, and therefore we will now examine them in greater detail.

9.2.1. Mathematic models of vortices

We will now look at a mathematical description of the flow field during vortex flow, and we will define a model of vortices.

For the purpose of examining rotation flows, we will introduce the cylinder coordinate system, which can be characterized by coordinates \((r, \Theta, z)\), where \(r\) is a radius, \(\Theta\) is an angle and \(z\) is a coordinate along the axis of a vortex. The relevant unit vectors are \(e_r\), \(e_\Theta\) a \(e_z\) and vorticity is defined by the relationship

$$\omega = \nabla \times \mathbf{u} = \frac{1}{r} \begin{vmatrix} e_r & r e_\Theta & e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \Theta} & \frac{\partial}{\partial z} \\ u_r & r u_\Theta & u_z \end{vmatrix}. \quad (10.15)$$
However, the vorticity seen as a physical phenomenon has no direct connection to a rotation of the fluid region as a whole; it can be brought into existence by a pure shear of a fluid element.

Let us assume the planar flow of an inviscid fluid is characterized by the fact that the velocity distribution is only a function of coordinates $r$ and $\Theta$ and it is independent of $z$. Then it is possible to define the so called “potential” or sometimes also alternatively the “line vortex” using the expression

$$ u = \frac{\Gamma_0}{2\pi r} e_\Theta, $$

(10.16)

where $u$ is the vector of velocity and $\Gamma_0$ is the rotation of the vortex filament. This relationship follows from the Biot-Savart law as will be shown in the following chapter.

After substituting into the defining relationship for the vorticity (10.15) it becomes obvious that the vorticity is, despite the fact that the global rotation of the fluid takes place, equal to zero within the entire domain with the exception of the point $r = 0$, where both the vorticity and the velocity remain undefined ($u_\Theta$ assumes an infinite value at this point). Therefore, we can come to the conclusion that despite the fact that the fluid is circulating, the flow of the fluid remains non-turbulent within the entire domain, except the on the rotational axis. The particle elements of the fluid are moving along the circular paths, but there is no rotation of these elements around their respective axis and the fluid elements keep moving along in a uniform circular translational motion. Figure 10.2 (a) presents the case of a potential vortex where $u_\Theta \sim r$. If the angular velocity of the fluid rotation is $\Omega$, then the velocity of flow in a given point is

$$ u = \Omega r e_\Theta. $$

(10.17)

When calculating the vorticity we obtain the component in the direction of z coordinate equal to $2\Omega$, while the other coordinate components are zero.

\[ \text{Figure 10.2 – Potential vortex (a) and rotational (turbulent) flow (b)} \]

By blending the two examples of the rotation depicted in Figure 10.2 (a) and (b), we obtain the so called “Rankin vortex”, which represent the simplest model of a real vortex in a fluid. This type of vortex has is vorticity concentrated within its core of comparatively small proportions and outside this core the vorticity remains practically zero. The Rankin vortex assumes that the core of diameter $a$ revolves as a rigid body while the rest of the flow field behaves as the potential vortex model does. The velocity profile is therefore given by the following relationships:

$$ u_\Theta = \begin{cases} \Omega r, & r < a, \\ \frac{\Omega a^2}{r}, & r > a, \end{cases} $$

$$ u_r = u_z = 0. $$

(10.18)
The shape of the circumferential component of the velocity and vorticity in the Rankin vortex is depicted in the Figure 10.3. The vorticity \( \omega_z \) is nonzero and constant in the core of the vortex and outside the core is equal to zero.

![Figure 10.3 – Rankin vortex – circumferential velocity and vorticity](image)

All this applies only for the flow of an inviscid fluid. Let us turn our attention to the vorticity and the process of vortex diffusion in a real viscous fluid. Let us tackle the case of vortex filament characterized by the rotation \( \Gamma_0 \) at the time \( t_0 \) within a viscous fluid having a kinematic viscosity equal to \( \nu \). From the N-S equation for vorticity of this type of flow follows the relationship expressing the rotation \( \Gamma(r,t) = 2\pi u_\theta(r,t) \) in cylindrical coordinates of the following form:

\[
\frac{\partial \Gamma}{\partial t} = \nu \left( \frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) \tag{10.19}
\]

with the initial condition

\[
\Gamma(r,0) = \Gamma_0 \tag{10.20}
\]

We demand only the bounded values of \( u_\theta \) at any given time and therefore the following must hold:

\[
\Gamma(0,t) = 0, \quad t > 0 \tag{10.21}
\]

The solution of the equation (10.19) can be found in the following form

\[
\Gamma = \Gamma_0 \left( 1 - e^{-r^2/(4\pi \nu t)} \right) \tag{10.22}
\]

and therefore also

\[
u_\theta = \frac{\Gamma_0}{2\pi r} \left( 1 - e^{-r^2/(4\pi \nu t)} \right). \tag{10.23}
\]

Then the vorticity is governed by the following relationship:

\[
\omega_z = \frac{\Gamma_0}{4\pi \nu t} e^{-r^2/(4\pi \nu t)}. \tag{10.24}
\]

An analysis of these results demonstrates that for distances from the \( r \) axis greater than \( \sqrt{4\pi \nu t} \) the significant influence on the value of rotation cannot be expected because virtually no vorticity managed to propagate into this region. On the other hand, for the \( r \) much smaller, the flow of fluid cannot be considered turbulence-free, because

\[
u_\theta \approx \frac{\Gamma_0}{4\pi \nu t} r \text{ for } r \ll \sqrt{4\pi \nu t}, \tag{10.25}
\]

this relationship depicts the fluid rotation as the rotation of a rigid body with the velocity \( \Gamma_0/8\pi \nu t \). The intensity of the vortex is therefore diminishing with a time progress, while the diameter of the vortex core becomes larger. Thus is defined the so-called “Lamb-Oseen model” of a vortex. The Figure 10.4 indicates the distribution of vorticity and circumferential velocity as they depend on the radius \( r \) and time \( t \). For the temporal data in these graphs the relationships \( t_1 < t_2 < t_3 \) apply and the points on the curves signify the value \( r = \sqrt{4\pi \nu t} \).
To study the phenomenon of vortices’ stretching the spatial aspect of the fluid flow must be taken into consideration as the planar models of vortices are not good enough to tell us anything about 3D behavior of vortices.

Therefore, let us turn our attention to a 3D model of a vortex, where its core is longitudinally stretched as shown in Figure 10.5. This is called the “Burgers vortex”, which models the convection of the vortex, its diffusion as well as its longitudinal stretching. Analytical solution of N-S equation yields the following velocity components expressed in the cylindrical coordinate system

\[
\begin{align*}
  u_r &= -\frac{1}{2} \alpha r, \\
  u_\theta &= \frac{\Gamma}{2\pi r} \left(1 - e^{-\alpha r^2/4\nu}\right), \\
  u_z &= \alpha z,
\end{align*}
\]

where \( \alpha > 0 \) and \( \Gamma \) are constants. The circumferential velocity is qualitatively indicated in Figure 10.5 (cf. Figure 10.2 applicable to the inviscid fluid).

The following applies to the vorticity:

\[
\omega = \frac{\alpha \Gamma}{4\pi \nu} e^{-ar^2/4\nu} e_z,
\]

It is obvious that the vorticity is concentrated in the core of the vortex having the diameter of about \( \sqrt{\alpha/\nu} \).

Burgers model of a vortex represents the real vortex well in an unbounded space, where there is no link between the secondary flow (stretching of the vortex) and its intensity.
9.2.2. Biot-Savart law

Let us consider the straight vortex filament in the fluid. The presence of the vortex filament induces within the fluid a velocity field in a similar way as was in the case of potential vortex (cf. above). In the three-dimensional case according to the Biot-Savart law\textsuperscript{11} the velocity vector \( u_i \) is induced in the given point in space perpendicular to the plane defined by the vortex filament element \( ds \) and by the radius-vector \( r \) determining the mutual position of the of the point in space and the vortex filament element. The magnitude of the velocity is determined by the intensity of the vortex filament expressed by means of the rotation \( \Gamma \) and it is inversely proportional to the square\textsuperscript{12} of a distance of the point from the line passing through the vortex element.

\[
    u_i = \frac{\Gamma}{4\pi} \int \frac{1}{r^3} ds \times r .
\]  

(10.28)

The situation is schematically indicated in Figure 10.6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10_6.png}
\caption{Induced velocity according to Biot-Savart law}
\end{figure}

Take note that the equation (10.28) does not contain any reference to the viscosity. The mechanism of inception of induced velocity in the vicinity of the vortex filament therefore does not relate to viscosity effects and thus the potential vortex can theoretically pop up even in the hypothetical inviscid fluid.

Simple examples of planar vortices are known where, for example, two vortices of the same rotation orientation are orbiting each other about the center of gravity of their connecting line and the two vortices with mutually opposite sense of their rotations and of the same intensity move with a uniform velocity along a straight line connecting both vortices. Unfortunately, such trivial cases do not exist in our three-dimensional world.

The straight line filament in the three-dimensional fluid generates along its length a zero velocity. This is due to a circumstance that the vectors of all filament elements are parallel to all position vectors of the element’s points. In the case of a curved or otherwise deformed vortex filament or in the case of more vortex filaments being present, the induction of individual elements’ velocity takes place and thus the filament begins to move and deforms itself. The straight line vortex filament in 3D space obviously represents an unstable configuration. The slightest disturbance of its shape triggers non-zero value of induced velocity and thus the further deformation of the filament.

\textsuperscript{11} Biot-Savart law is better known in the theory of electricity and magnetism where it determines the force acting on a conductor in magnetic field; formally the expression is the same but the meaning of the individual entities involved is different.

\textsuperscript{12} In the formula (10.28) one \( r \) is cancelled out.
The interaction of spatial vortex filaments is a powerful dynamical and quite complex process. Each element of the vortex filament induces velocity in locations of all other elements, which are not collinear with the element in question and thus alters the shape of the filament as the whole.

### 9.2.3. Mutual interaction of vortices

According to the Thomson theorem on vortices the modification of vortex filaments due to their mutual interaction is not possible; however, under real conditions, due to viscosity the modifications do take place. The two distinct cases can be observed, namely those of vortices having the same orientation of their rotation and those having their rotation in mutually opposite senses.

“Merging” or “pairing” of vortices can be observed in the cases of planar vortices rotating in the same direction. Practically the phenomenon may take place in a flow through highly anisotropic regions (e.g. in thin layers of the flow fluid) or in regions where the parallel vortices were generated artificially, e.g. in planar shear layers. Both vortices have to rotate in the same direction. After starting the process it is necessary for both vortices to have a sufficient intensity and for their distance to be less than the critical one. Then the merging of the two vortices takes place and the fluid from both vortices becomes mixed. The temporal sequence of this process (the planar one) is indicated in Figure 10.7.

\[\text{Figure 10.7 – Connecting of the two vortices of the same orientation}\]

Considering the interaction of the two parallel filaments, this process does not evolve the same way in all sections perpendicular to the axis of vertex direction; we record significant spatial effects related to the instability of the straight shape of the vortex filament. Figure 10.8 indicates the deformation of vortex tubes during the initial phase of merging. The pseudo-periodic spatial disturbances are clearly visible and the whole process becomes more and more complicated by disturbances of ever higher order.
This phenomenon is remarkable due to the fact that in this case the direction of energy flow is just opposite to the classical cascade transfer of energy, where the energy is transferred from the large vortices to the small ones. The pairing mechanism represents the basic mechanism of the inverted cascade (cf. 7.3.2), when the energy of small vortices is, by their merging, transferred to the larger ones.

In spatial configuration of vortex filaments the interaction between them can take place due to viscosity. They keep disengaging and reuniting themselves. This phenomenon is known as reconnection or also as bridging. The process takes place in several phases and is, due to its very substance, the three-dimensional one. At first, due to instability, the local approach of certain parts of the filaments takes place resulting in their mutual contact at a certain point with the passage of time. This leads to interruption of the filament at the contact point due to viscosity effects. In the concluding phase of the phenomenon the filament separate again, but in different order. Thus the vortex loops come into existence. The temporal development of the individual phases of the process is shown in Figure 10.9.

The process is, in case of sufficiently long vortices almost periodic in the surrounding space. The parallel straight-line vortices thus develop themselves into vortex circles as shown in Figure 10.10 where the individual phases are depicted in chronological order from left to right.
9.2.4. **Mechanism of vorticity generation**

In Kolmogorov theory of fully developed isotropic turbulence, the cascade transfer of energy from the larger vortices to the smaller ones is enshrined. Kolmogorov does not elaborate the physical basis of this phenomenon. As to the vorticity, we have the Helmholtz equation, which was derived from the N-S equations in Chapter 4:

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega + (\omega \cdot \nabla) \mathbf{u}. \quad (10.29)$$

The physical meaning of the individual terms of this formula is quite clear: the term on the left side of the equation represents the temporal variation of vorticity, the first term on the right-hand side is viscous diffusion and the second one represents the production of vorticity.

Let us examine in greater detail the term representing production of vorticity \((\omega \cdot \nabla) \mathbf{u}\). The vortex line, i.e. the vector line of the vorticity field is drawn in Figure 10.11. In the point of the vortex line under examination we have erected the rectangular coordinate system having the unit vectors \(\mathbf{e}_s\) in the direction of vortex line and perpendicular vectors \(\mathbf{e}_n\) and \(\mathbf{e}_m\) (the vorticity has no component into these directions). Then the production term form the equation (4.40) assumes the following form:

$$\left(\omega \cdot \nabla\right) \mathbf{u} = \left[\omega \cdot \left(\mathbf{e}_s \frac{\partial}{\partial s} + \mathbf{e}_m \frac{\partial}{\partial m} + \mathbf{e}_n \frac{\partial}{\partial n}\right)\right] \mathbf{u} = \omega \frac{\partial \mathbf{u}}{\partial s}. \quad (10.30)$$

Here \(\omega\) is the modulus of the vorticity vector. If, for this moment, we neglect the influence of viscosity, then the further development of the vortex characterized by the vortex line in Figure 10.11 will take place according to the equation

$$\frac{D\omega}{Dt} = \omega \frac{\partial \mathbf{u}}{\partial s}. \quad (10.31)$$

This equation can be divided into three components. The component in the direction of \(\mathbf{e}_s\) \(\frac{D\omega}{Dt} = \omega \frac{\partial \mathbf{u}}{\partial s}\) represents stretching of vortex filament and the components \(\frac{D\omega}{Dt} = \omega \frac{\partial \mathbf{u}_m}{\partial s}\) and \(\frac{D\omega}{Dt} = \omega \frac{\partial \mathbf{u}_n}{\partial s}\) represent its sideways inclination.
Now, let us consider the special case of planar velocity field, i.e. the field where one component of the velocity is zero in the entire region: $u_3 = 0$ and the components $u_1$ and $u_2$ are, generally speaking nonzero. Then obviously holds that the vorticity vector can have as a nonzero component only the component $\omega_3$, while the other components are always equal to zero: $\omega_1 = \omega_2 = 0$. However, under these conditions the $\omega \cdot \nabla = 0$ identity holds and the production of vorticity is therefore zero. It is clear that in planar flow the mechanism of vorticity, which is typical for the turbulent flow, cannot exist. Hence we can derive the general conclusion that the turbulent flow must be always the spatial one.

The stretching and inclination of vortices describes the process of interaction of vorticity and velocity gradients in space. This process begins as a structure defined in one direction within the space, by the stretching and inclining of the vortex in the plane perpendicular to its axis is subsequently generated vorticity, which again interacts with the velocity field and thus creates the new vorticity. Thus the velocity and vorticity fields become gradually more and more chaotic and randomized, the turbulence begins to appear and remains supported in its existence by this process. Smaller and smaller vortices are generated, their distribution becomes more and more isotropic and thus the information about the original orientation of the primary large-sized structure is lost. This situation is schematically illustrated in Figure 10.12. We start from one energetic vortex oriented in the direction of $x_1$ axis. During the rotation of this vortex, according to the Biot-Savart law, in the vicinity of this vortex the induction of velocity in the plane perpendicular to its axis $(x_2, x_3)$ can take place and therefore the energy can be transferred to smaller eddies in directions $x_2$ and $x_3$. This process continues toward ever smaller vortices: after passing through 4 levels we have almost homogenous distribution of the vortices’ orientation: direction $x_1$ 6 off, direction $x_2$ 5 off and direction $x_3$ 5 off. In reality, the orientation of the individual vortices does not necessarily have to be identical with the orientation of the axis of the coordinate system; moreover, the vortices are spatially deformed and therefore the process of homogenization is even faster than that in the schematic of Figure 10.12.
Now, let us examine the vortex tube and try to determine its temporal development. According to the second Helmholtz theorem (cf. Chapter 10.1) the vortex tube moves together with the fluid as if being a material object. Furthermore, according to Kelvin theorem its intensity does not change with the passage of time. The fluid is incompressible, its volume between two transversal sections remains constant (the situation is schematically illustrated in Figure 10.13). Therefore, during stretching of the vortex tube its cross section becomes smaller and thus the corresponding vorticity increases because the value of the circulation remains unchanged. Therefore, it can be stated that stretching of the vortex tube brings about an increase of the absolute value of the vorticity of the vortex filament in its core. This mechanism of vorticity generation is called vortex stretching.

Ideas similar to those applicable to the vortex tube are applicable also to the isolated vortex line. The vortex line can also stretch itself, but it does not change its cross section during the process (its cross section approaches 0, it is a one-dimensional structure). Here the value of vorticity is bound to the length element of the vortex filament in being tangential to this element.
and its magnitude is proportional to the length of the element. The constant of proportionality is given for the given filament and its magnitude remains unchanged.

Stretching of the vortex filaments represents one of the principal mechanisms of vorticity generation and of Reynolds stress and therefore also of the turbulence per se.

9.2.5. Other forces acting upon vortex structures

Vortices in a flow fluid behave in the fluid to a great extent as a body of cylindrical shape with, generally speaking, a curved axis. These “bodies” keep changing their shape and position as being carried along by the flow fluid. This phenomenon is due to the fact that as per the Helmholtz theorem the vortex filaments and vortex tubes move as material objects.

Therefore, the parts of vortices are exposed to the drag forces dependent on the velocity of flow and the shape of the vortex. Another force acting on the rotating fluid is the Magnus force. This force has its origin in rotation of the fluid flow; its direction is perpendicular to the direction of rotation as well as to the direction of the flow. Therefore, this force has the characteristic of the buoyancy.

These forces may cause deformation of the vortex tube on one hand, and its motion on the other hand. The motion is directed at certain inclination with respect to the principal motion of the fluid. The trajectory of the rotating object has the shape of an arch. This effect is well known to most players of ball games: the rotating ball does not move along the “honest” trajectory. The situation is schematically illustrated in Figure 10.14.

![Figure 10.14 – Magnus force](image)

9.3. Self-sustaining mechanisms of turbulent flow

The well-developed (“mature”) turbulent flow has the innate ability to sustain itself, at least in a statistical sense of the word. In the text below we shall examine the mechanisms of such processes in a greater detail.

9.3.1. Coherent structures in wall flows

In turbulent boundary layers the process of turbulence generation is determined by the three kinds of quasi-periodic (or quasi-stochastic) vortex structures; they are large external structures, Falc events of mean magnitude and the events in close proximity to the wall.

The size of the large spatial external structures is determined by the size of the entire shear region, i.e. by the thickness of the boundary layer. These vortices direct the dynamics of the boundary layer in its external region; the process involved is dragging off and production of turbulence. They appear themselves randomly in space and time and for low Reynolds numbers
they are essentially remnants of Emmons turbulence stains originating at the phase of transition to the turbulent state.

Falc vortices are also highly coherent and spatial. They are vortices that originate in wakes, Emmons turbulent stains, and grid turbulence, and in boundary layers. These vortices are typically about 100 wall units large. The Falc vortices represent a link between the large structures and the structures in the vicinity of the wall.

The third kind of vortices exists in the region near the wall and it is responsible for the production of about one third of the Reynolds stress.

### 9.3.2. Hairpin vortices

The hairpin vortices are generally considered to represent the structures playing a role during the transfer of the boundary layer to turbulent state (cf. 6.3.1.) However, it turns out that these entities play an equally important role in the process of self-sustaining or regeneration of the fully developed turbulent boundary layer. The presence of the hairpin vortices of different sizes in the developed boundary layer was noticed by Theodorsen in 1952.

The fluid tends to stick to the wall and at nonzero velocity of the external flow the shear layer is originated in the close proximity to the wall. In an immediate proximity of the wall originates the vortex filament parallel to the wall and perpendicular to the direction of the main flow velocity due to Tollmien-Schlichting instability. This item is subject to a subsequent violent development. It converts itself into the “hairpin vortex” (sometimes also referred to as the *hairpin horseshoe vortex*). The vortex structures are carried by the flow. We can visualize the origin of these horseshoe entities as originating from a small disturbance of flow in the boundary layer in the shape of a vortex after expansion. The upper part of the horseshoe shape is dragged downstream and the arms are stretched and therefore they keep rotating faster. Velocities induced by this rotation of arms causes pushing of the upper part upwards into the region of greater velocity. This results in elongated arms forming the pairs of steaks.

Research of the coherent structures in turbulent flow is, generally speaking, a very difficult task because it is necessary to monitor the temporal topology variations. For a long time the researchers had been dependent on just qualitative visualization methods; at the present time the planar and also spatial experimental methods (PIV variants) are used. However, the most effective method for researching the turbulence structures is now the method of direct numerical simulation DNS. Most results to be presented further down in this text were obtained by this method.

![Hairpin horseshoe vortex](image)

*Figure 10.15 – Hairpin horseshoe vortex*

Robinson (1991) examined thoroughly the DNS results obtained by Spalard (1988). In Figure 10.15 the typical horseshoe shape of the vortex is schematically illustrated according to Robinson; it consists of a head, a neck and a leg. The horseshoe vortex in the picture is in its ideal state; however in the flow either the deformed structures or even the incomplete ones may exist. The regular shape originating during the natural transition into the turbulent state (cf. 6.3.1)
represents the exception rather than the rule. The horseshoe vortex is a dynamical entity with a rather violent development, which is determined by the interaction with the shear flow existing due to external flow and also due to interaction with the neighboring structures. Due to convection the arms (legs) stretch themselves and form the longitudinal vortices. These vortices play an important role in the mechanism of generation and sustaining of the turbulent structure within the boundary layer. The above described mechanism of the origin of longitudinal vortices represents just one of several other possibilities.

From the shape of the hairpin vortex it follows that the longitudinal vortices are located predominantly in the inner part of the boundary layer near the wall in what is called the buffer layer. On the other hand, near the external edge, the so-called wake, the transversal structures dominate. In between the two mentioned regions there is the region of logarithmic law; here we can see the mixture of both types of vortex structures. The situation in the turbulent boundary layer is depicted in Figure 10.16.

![Figure 10.16 – Types of coherent vortices in the turbulent boundary layer](image)

The hairpin vortices can arrange themselves into packets. The concept of packets of the hairpin vortices was proposed in 1999 by Adrian. Figure 10.17 illustrates schematically 3 packets of hairpin vortices. The light color indicates the vorticity of the hairpin vortices while the large dark stains represent regions of low velocity. The hairpin vortices are generated in packets and during the first stage they are held together while growing in size; later on, they move over to the region of external flow. In addition, Adrian observed the existence of comparatively large regions of almost constant velocity and he speculates that they are formed by the packets of organized vortices and that the number of vortices in one packet depends on the Reynolds number. The structure of these packets is a very complex one (fractal) as inside the larger packets there are the smaller ones. The results in Figure 10.17 correspond to the Reynolds number $Re_0 = 6845$; this value is generally considered large enough to ascertain the fully developed turbulent flow in the boundary layer. The size of the packet of the hairpin vortices is about $0.8 \delta$ in their height and $2 \delta$ longitudinally along the direction of flow ($\delta$ is the thickness of the boundary layer).
9.3.3. **Longitudinal steaks of low velocity and the “bursting phenomenon”**

The low velocity steaks play a very important role in the process of turbulence generation. They were identified near the wall and they are oriented in the direction of flow (cf. Figure 10.18.).

The production of the Reynolds stress is of intermittent nature. About half of the total turbulent kinetic energy takes place within the first 5% of the boundary layer at the typical Reynolds numbers attainable in experimental laboratory conditions. This process is called the “bursting phenomenon”. The process begins as an elongated vortex pair rotating in opposite directions to each other that is oriented in the direction of the main flow and that has the diameter of about 40\(r/u_\tau\). These vortices are influenced by a strong shear and they induce regions of low and high velocity in between themselves, cf. Figure 10.18. The vortices and other structures appear in the
space and time quite randomly but nevertheless, pseudo-periodically; the mean wavelength in the transversal direction is about 80 through $100v/u_\tau$, as was observed for the first time by Kline (1967). Kline also recorded that the regions of low velocity develop themselves downstream and the profiles of mean velocities with inflection point, which are unstable even according to the inviscid theory, are generated. In this region oscillations appear between the domains of low and high velocity and this phenomenon represents a sign of the origin of the secondary instability. The regions of low velocity are subsequently pushed away from the wall while the amplitude of the oscillations keeps growing; a sudden “jump” to the turbulent state follows. This process is very fast and therefore Kline refers to it as the “burst”. Corino and Brodkey (1969) demonstrated, that the regions of low velocity are comparatively narrow (approx. $20v/u_\tau$) and they can have also a transversal component of not quite a negligible dimension. Practically the entire production of the kinetic turbulent energy in the region close to the wall originates by means of this mechanism.

The importance of the streaks of low velocity was pointed out already Kline in 1967. The term streak is based on the method of visualizing turbulence by means of smoke; the smoke keeps concentrating in the regions of low velocities. More generally, the streaks are understood as the region of flow with velocity oscillations in transversal direction; the component of vorticity in the direction of the main flow is negligibly small in this region. The existence of streaks has important influence on the stability of flow within the boundary layer. Kline observed that smoke gradually separates from the wall, begins to oscillate and suddenly “explodes”; Kline christened this phenomenon the bursting phenomenon – cf. Figure 10.19 (bursting phenomenon was described for the first time by Klebanoff (1962)).

![Diagram](image)

**Figure 10.19 – Kline representation of the bursting phenomenon**

Corino and Brodkey (1969) supplemented Kline representation by the last phase, namely by assuming the penetration of fluid from the outside of the region. Thus, the idea of the bursting phenomenon was coined; it assumed the form of a pair of coherent events, the “ejection” and the “sweep”. According to estimates, the bursting process is a mechanism, representing production of Reynolds tension to the tune of up to 80%. Bogard and Tiedermann (1987) demonstrated a bit later that one “burst” typically contains several phases of ejection. Individual hairpin vortices can generate individual events of “ejection”, while the packet of hairpin vortices (cf. Figure 10.17) generates the whole series of these events, being typical for the bursting phenomenon. Velocity fluctuations in the direction of the main flow $u$ and that of perpendicular to the wall $v$ can be visualized in the graph presented by Figure 10.20 divided into 4 quadrants. States illustrated in Q2 and Q4 quadrants represent production of Reynolds tension. Q2 can be interpreted as a sweep event while Q4 represent an election event.
The bursting phenomenon was examined more thoroughly only in recent time. Falco (1991) demonstrated that the typical vortex existing due to pushing of fluid away from the wall and which moves against the wall induces a strong event of sweep (penetration) characterized by its high value of $uv$ ($u$ positive, $v$ negative). The region next to the wall is incessantly bombarded by packets of fluid with very high velocity, which originated in either logarithmic or external regions. This results in a tendency to bring about and enhance the inflection nature of the velocity profile by strengthening of the momentary shear. Thus the origin and growth of instabilities is supported.

9.4. **Dynamics of the coherent structures**

By their nature, the coherent structures are dynamical objects having their specific origin, temporal development, mutual interactions and final disappearance or conversion to something else.

9.4.1. **Origin of coherent structures**

The process giving rise to coherent structures was examined, using DNS methods, by Adrian and Zhou in their series of articles (1988 – 1999). As a point of departure the authors used a typical turbulent velocity profile, into which they placed a typical vortex Q2 (this vortex was experimentally identified by Kim at al. in 1987); the vortex in question is essentially a vortex pair, cf. Figure 10.21 (a). The vortices are about 200 wall units long, the distance of their ends from the wall is 12 at the ends pointing upstream and 5 at the ends pointing downstream. In the transversal direction the typical distance of these structures is 100 units – this is the typical distance of vortex streaks. Figure 10.21 (a)-(d) illustrates typical development of this structure: approximately at $2/3$ of its length the “bridge” appears, and finally the typical hairpin vortex develops. However, this scenario applies only at a sufficient intensity of initial vortices; in the case of weaker vortices the damping takes place and finally the structures entirely dissipate themselves.
The following Figure 10.22 illustrates the later time development of the hairpin vortex. The original hairpin vortex, gives rise to secondary and tertiary vortices upstream from the head of the original hairpin vortex. Later on the interconnection of rear arms of the vortex pointing downstream takes place and there appears another head of a hairpin vortex further downstream (DVD). Other smaller and weaker structures keep appearing, but these dissipate quite rapidly. The entire process results into an appearance of the packet of hairpin vortices cf. Figure 10.17.
The above described mechanisms work fine even if one vortex from the pair significantly differs from the other; of course, in such case all the resulting structures are quite asymmetrical. Asymmetry may grow to such extent that finally the complete degeneration of one of the arms of the hairpin vortex takes place. The two principal mechanisms giving rise to new vortices can be attributed to this. One is the bridging of the two parallel longitudinal vortices, i.e. the appearance of the new vortex oriented across the fluid low, while the other process generates new longitudinal vortices due to the influence of already existing and sufficiently strong longitudinal vortices – i.e. vortex regeneration.
9.4.2. Regeneration of coherent structures

Coherent structures are imbued with the ability to regenerate themselves; during the development of the parental generation of hairpin vortices the new descendant generation of longitudinal vortex pairs of the type Q2 begins to appear. The mechanism of an emergence of the new Q2 event owes its existence to the interaction of the parental generation with the fluid flow and with the wall. The regeneration process was studied by Brooke and Hanratty (1993). They found out that the new vortex develops in the region under the end of the parental vortex pointing downstream. The end of the parental vortex moves away from the wall and thus a local peel-off takes place. The orientation of the vorticity in the new vortex is opposite to that of the parental vortex. The entire process takes place inside the inner region in close proximity to the wall and therefore it is not significantly influenced by the event taking place in the outer region. This scenario has been confirmed by numerous studies based on the DNS method.

The condition for a successful process of regeneration of the longitudinal vortex is its sufficient strength and it must be located close to the wall. The new vortex appears at the side of the old vortex where the flow is directed towards the wall (sweep); it gradually creeps under the parental vortex and sometimes it actually pushes it away from the wall. The new vortex has a sufficiently strong vorticity component pointing perpendicularly towards the wall; by the action of shear the vortex is straightened up and begins to orient itself in the downstream direction. During all of this metamorphosis the vortex becomes longer and stronger.

According to Smith the process of regeneration is closely related to the phenomenon of their nesting, as indicated in Figure 10.23. During the nesting process the arms of the vortices arranged in lines one after another become entangled and ultimately mutually merge together. A somewhat different concept of the regeneration process was proposed by Robinson (1991). He assumes only an incomplete hairpin vortex having only one arm fully developed. The head of the new small hairpin vortex (“new arch”) begins to appear in the region of the longitudinal stripe of low velocity, located behind the developed arm of the hairpin vortex. Thus, the new hairpin vortex may come into existence even behind the solitary longitudinal vortex of a different origin, cf. Figure 10.24.

Figure 10.23 – Entanglement of hairpin vortices
Other mechanisms of creation of hairpin vortices have been described. For example, Smith (1984) observed the creation of a new vortex by the action of the head of the hairpin vortex, i.e. due to vorticity oriented perpendicularly to the direction of the main flow. This represents a very important mechanism known as the viscous/inviscid interaction (viscous – inviscid interaction). This phenomenon was described in detail by Doligalski, Walker (1984), for example. Here we deal with the interaction of the transversal vortex perpendicular to the main fluid flow and parallel to the wall, which is strong enough to allow neglecting of the impact of viscousity together with the shear layer near the wall. The vortex moves in parallel with the wall. Due to velocity induced by the vortex the local reduction of the velocity takes place in the close proximity of the wall; in the region next to the moving vortex the velocity may be reduced to zero and the boundary layer peels off and the new vortex of opposite vorticity orientation is generated. This process is sometimes referred to as the eruption process. The situation is depicted in Figure 10.25, where the original vortex (parent vortex) and the new one (child vortex) are illustrated. The main fluid flow goes from right to left.

![Figure 10.24 – Regeneration of the incomplete hairpin vortex](image)

*Figure 10.24 – Regeneration of the incomplete hairpin vortex*

![Figure 10.25 – Emergence of the new vortex due to viscous/inviscid interaction](image)

*Figure 10.25 – Emergence of the new vortex due to viscous/inviscid interaction*
10. Index of References

10.1. Literature Recommended for Further Reading


10.2. References

(Short List)
Gad-el-Hak, M., 2000, Flow Control; passive, active, and reactive flow management. Cambridge University Press.
Reynolds, O., 1883, Phil.Trans.R.Soc.Lond., 174, 935-982.
10.3. List of Third Party Illustrations


Figure 7.5 Saddoughi, S. G., S. V. Veeravalli, 1994, Local isotropy in turbulent boundary layers at high Reynolds number. J.Fluid Mech. 268, 333-372.

Figure 10.7 Buntine, J.D., Pullin, D.I., 1989, Margin and cancellation of strained vortices. J.Fluid Mech., vol.205, 263-295.


Figure 10.17 Adrian, R.J., Meinhart, C.D., Tomkins, C.D., 1999, Vortex organization in the outer region of the turbulent boundary layer. J Fluid Mech; 422, 1-54.

Figure 10.18 Blackwelder, R.F., 1978, The bursting process in turbulent boundary layers. eds. C.R.Smith and D.E.Abbott, 211-227, Lehigh University, Bethlehem.


Figure 10.22 Zhou, J., Adrian, R.J., Balachandar, S., 1996, Autogeneration of near-wall vortical structures in a channel flow. Phys Fluids, vol8, 288.

Figure 10.23 Smith, C.R., 1984, A synthesized model of the near-wall behaviour in turbulent boundary layers. In: Patterson GK, Zakin JK, editors. Proceedings of the Eighth Symposium on Turbulence, University of Missouri Rolla.

Figure 10.25 Panton, R.L., 2001, Overview of the self-sustaining mechanisms of wall turbulence. Progress in Aerospace Sciences 382 37, 341-383.


Figure 12.6 Sigurdson, L.W., 1997, Flow visualization in turbulent large-scale structure research. in Atlas of Visualization III, edited by The Visualization Society of Japan, 99–113, CRC Press.

Other illustrations come from the author’s archives and from freely accessible Internet sources.
11. Annexes

11.1. Vector calculus

In this text the following two representations of vector calculus are used: the vector symbolism and the components symbolism. Both of these systems of representation are mathematically equivalent. When using the component representation, we shall make use of Einstein summation rule, i.e. the repeated index is the one over which the summation must be performed.

Let \( \mathbf{a} \) and \( \mathbf{b} \) (\( a_i \) and \( b_i \)) be arbitrary vectors and \( c \) and \( d \) be arbitrary scalars.

Here is the scalar and vector product definition:

The scalar product of the two vectors is defined as follows:

\[
\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = a_1b_1 = \delta_{ij}a_i b_j,
\]

where \( \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \) is the Kronecker delta.

Vector product of the two vectors is defined as follows:

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2),
\]

\[
= a_i b_j - a_j b_i = \varepsilon_{ijk} a_i b_j,
\]

where \( \varepsilon_{ijk} \) is the Levi-Civita alternating tensor and \( \varepsilon_{ijk} \) is the Kronecker delta.

The symbol \( \nabla \) is the Laplace operator in partial derivatives of the following form:

\[
\nabla = \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}.
\]

Divergence of the vector is given as follows:

\[
\text{div} \mathbf{a} = \nabla \cdot \mathbf{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}.
\]

Divergence of the vector differs from its gradient, which is expressed as follows:

\[
\text{grad} \mathbf{a} = \nabla \mathbf{a} = \begin{pmatrix} \frac{\partial a_1}{\partial x_1} \frac{\partial a_2}{\partial x_1} \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_1}{\partial x_2} \frac{\partial a_2}{\partial x_2} \frac{\partial a_3}{\partial x_2} \\ \frac{\partial a_1}{\partial x_3} \frac{\partial a_2}{\partial x_3} \frac{\partial a_3}{\partial x_3} \end{pmatrix} = \frac{\partial a_j}{\partial x_j}.
\]

Finally, the rotation of the vector is defined in this way:
From the theory of tensor calculus follow certain vector related identities of general validity:

\[ \varepsilon_{ijl} \delta_{lm} - \delta_{jm} \varepsilon_{il} = 0, \]

\[ \nabla \times \nabla \phi = 0, \]

\[ (d \nabla \phi) \times \nabla \phi = 0, \]

\[ \nabla \cdot (\nabla \times a) = 0, \]

\[ a \cdot \nabla = (\nabla \times a) \times a + 1/2 \nabla (a \cdot a), \]

\[ \nabla \times \nabla^2 a = \nabla^2 (\nabla \times a), \]

\[ \nabla \times (a \times b) = a (\nabla \cdot b) + (b \cdot \nabla) a - b (\nabla \cdot a) - (a \cdot \nabla) b. \]

\[ 12. \quad \text{Quantities used in turbulence theory} \]

Below follows the overview of principal quantities frequently used in the theory of turbulence. The quantities presented here are all “specific” quantities, i.e. they are related to a unit of mass of the fluid under study.

“Kinetic energy” is given by the following formula

\[ k = \frac{1}{2} u_i u_i. \]

“Rate of dissipation” is given by the following formula

\[ \varepsilon = 2 \nu s_{ij} s_{ij}, \]

where

\[ s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]

is the strain rate tensor indicating rate of deformation of the fluid particle.

Often used is the “specific rate of dissipation” given by the following formula

\[ \omega = \varepsilon/k. \]

“Vorticity” is the vector

\[ \omega = \text{rot } u, \]

and it represents the quantity reflecting the amount of rotational or circulation content of the fluid in a given point (to be more specific, it represents the amount of local angular velocity). The modulus (absolute size of the vector) of vorticity corresponds to the following circulation \( \Gamma \):

\[ |\omega| = d\Gamma/dS, \]

where \( S \) is the plane perpendicular to the vorticity vector.

Another important quantity is “enstrophy”, given as follows

\[ \zeta = \frac{1}{2} \langle \omega^2 \rangle. \]

Formally the enstrophy is similar to the kinetic energy, just instead of velocity the vorticity enters the relationship.

“Helicity” is given by the following formula

\[ H = \frac{1}{2} \langle u \cdot \omega \rangle. \]

It measures the helical content of a motion; for pure circular and translation motions of a fluid the helical content is zero.
11.3. Symmetry of the turbulent flow

In the theory of turbulence the number of assumptions is adopted in order to simplify the formal mathematical descriptions thereof. Below, some of these assumptions are presented and defined.

“Homogeneity”
Def.: Turbulence is homogenous if all centralized quantities at arbitrary \( n \) points \( x_1, x_2, \ldots, x_n \) (at time instances \( t_1, t_2, \ldots, t_n \)) are invariant with respect to any translation of these points of coordinate system in space by a vector \( \mathbf{r} \).
Hence follows: \( \langle u(x, t) \rangle = \langle u(x + \mathbf{r}, t) \rangle \), can represent only homogeneity of fluctuations, not of mean values; homogeneity of instant values is not considered.

“Stationarity”
Def.: Turbulence is stationary if all the mean values centered in any \( n \) time instances \( t_1, t_2, \ldots, t_n \) are invariant with respect to any shift of these time instances by a time interval \( \tau \).
Hence follows: \( \langle u_{\alpha 1}(x_1, t_1) \rangle \ldots \langle u_{\alpha n}(x_n, t_n) \rangle = \langle u_{\alpha 1}(x_1, t_1 + \tau) \rangle \ldots \langle u_{\alpha n}(x_n, t_n + \tau) \rangle \), energy input is necessary, if nonexistent, the damping due to dissipation occurs.

“Isotropy”
Def.: Homogenous turbulence is isotropic if all mean values at any \( n \) points \( x_1, x_2, \ldots, x_n \) (at any time instances \( t_1, t_2, \ldots, t_n \)) are invariant with respect to any rotation of these points or rotation of the coordinate system.
Hence follows: \( \langle u(x, t) \rangle = \mathbf{a} \Rightarrow \langle \mathbf{g}(x, t) u(x, t) \rangle = \mathbf{a} \) holds for any scalar \( \mathbf{g}(x, t) \).
Isotropy can be also considered in relation to higher statistical moments rather than to the moment of the 1st order (mean value).

“Ergodicity”
Average implementation in a sense of averaging of a set of values is substituted by spatial or by temporal average.

11.4. Statistical tools

The quantities characterizing the turbulent flow field at a given time and point can be considered to be stochastic quantities. Then the application of statistical tolls is fully justified.

11.4.1. Averaging

The averaging operation defined by Reynolds is to be understood as averaging of the set of quantities, i.e. making ensemble average from quantities obtained due to repeated implementation of a process in question. The mean value of the set of quantities of a variable \( a \) will be denoted by the symbol \( \langle a \rangle \). If we accept the hypothesis of the process ergodicity, then we can substitute the operation of averaging of the set of operation values by operation of averaging over a time interval yielding \( \bar{a} \). The mean time average of a quantity substituted by the average of their implementation over a certain limited period of time \( T \) yields the estimator of the quantity \( a_{\tau}(t) \)

\[
a_{\tau}(t) = \frac{1}{T} \int_0^T a(t + \tau) d\tau . \tag{12.21}
\]

Take notice that the estimator is, in general, the function of time. The mean value is given as

\[
\bar{a} = \lim_{T \to \infty} a_{\tau}(t) , \tag{12.22}
\]
and this limit is no longer time dependent.
In practical applications we always work with estimators that have their values close to the corresponding mean value. On the other hand, in theoretical considerations we most often work with mean values of the set of quantities. The time interval \( T \) over which we take the corresponding integral is of decisive importance here. In general it must be much longer than the period of the slowest quasi-periodic component of a process under study.

The averaging operation has several more or less trivial properties that determine the mode of its use.

The averaging operation is linear. For any two arbitrary functions and \( a(x,t) \) and \( b(x,t) \) and the constant \( \lambda \) holds the following relationship

\[
a + \lambda b = \bar{a} + \lambda \bar{b},
\]

(12.23)

commutativity of a derivative:

\[
\frac{\partial a}{\partial x} = \frac{\partial \bar{a}}{\partial x},
\]

(12.24)

and also the commutativity of integration

\[
\int a \, dx = \int \bar{a} \, dx,
\]

(12.25)

the double averaging rule

\[
\int a \, dx = \bar{a} \int dx,
\]

(12.26)

and finally, averaging of multiplication

\[
\int a \cdot \bar{b} = \bar{a} \cdot \bar{b}.
\]

(12.27)

Assumption of ergodicity of the process is usually adopted because it makes possible substitution of averaging of a set by averaging over time or space. Practically the process can be implemented by filtering the signal using the respective filter. It is applicable only to homogeneous and stationary turbulence.

### 11.4.2. Characteristics of the stochastic process

Let \( u \) be a physical quantity, which may assume random values \( v \). Below, we describe the statistical characteristics that can describe the manner of manifestations of this quantity in general (on average, so to speak).

#### 11.4.3. Distribution function and probability density

The nature of a stochastic process can be examined using a number of different tools. One of the principal characteristics of the stochastic process is the “distribution function (more precisely cumulative distribution function – CDF) \( F \). It is defined as the probability, that the specific manifestation of the stochastic process \( u \) shall be smaller than the value \( v \)

\[
F(v) \equiv P\{u < v\}. 
\]

(12.28)

From the definition of the distribution function it is obvious that the function is either rising or constant and that the following relationships hold:

\[
F(-\infty) = 0 \quad \text{and} \quad F(\infty) = 1.
\]

(12.29)

The “probability density” \( f \) (or more exactly probability density function – PDF) is defined, following Radon-Nikodym as:

\[
f(v) \equiv \frac{dF(v)}{dv}.
\]

(12.30)

From the definition of the probability density follow the following trivial properties thereof:

\[
\int_{-\infty}^{\infty} f(v) \, dv = 1, \quad f(-\infty) = f(\infty) = 0.
\]

(12.31)

The probability that the quantity \( u \) assumes the values within the interval \([v_a, v_b]\) can be expressed using the distribution function or the probability density function in the following manner:
\[ P \{ v_a \leq u < v_b \} = F(v_b) - F(v_a) = \int_{v_a}^{v_b} f(v) \, dv. \]  

(12.32)

Variability of a stochastic quantity is characterized in its entirety by the distribution function or by the probability density function; the information contents of both functions are identical. However, these functions do not say anything about whatever variations of the process may have occurred in time.

Experimental determination of the probability density or of the distribution function is rather problematic, because both functions in question are, in general, continuous functions. Probability density is usually presented as a histogram whereby the definition range of the function is divided into equal intervals. Selection of the size of histogram “step” is not arbitrary. It is necessary to take into consideration the number of samples falling into each histogram step. If the number of such samples is small, the appearance of the whole histogram remains haphazard and histogram information contents are worthless.

### 11.4.4. Some types of stochastic distributions

The uniform distribution is defined as follows:

\[ f(v) = \begin{cases} 
\frac{1}{b-a}, & \text{for } a \leq v < b, \\
0, & \text{for } v < a \text{ and } v \geq b. 
\end{cases} \]  

(12.33)

This distribution represents the basic statistical distribution.

The normal or Gauss distribution is characterized by its mean value \( \mu \) and by the variance \( \sigma^2 \)

\[ f(v) = N(v, \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(v-\mu)^2}{2\sigma^2} \right]. \]  

(12.34)

The normal distribution is typical for the purely stochastic process. It is quite convenient for analytical work, because both derivative and integral of the function can be easily found. Skewness factor \( S = 0 \) and flatness factor \( F = 3 \).

Many random quantities playing a role in turbulence have the log-normal distribution. Let the variable \( \tilde{v} \) be random and have the normal distribution. Then the variable \( v = e^{\tilde{v}} \) is characterized by the log-normal distribution:

\[ f(v) = \frac{1}{v \sigma \sqrt{2\pi}} \exp \left[ -\frac{(\ln v - \mu)^2}{2\sigma^2} \right]. \]  

(12.35)

The log-normal distribution is defined only for positive values of the random variable, i.e. quantities like distances, volumes and others may be subject to this mathematical description. Theoretical derivation of this distribution has its origin in consideration of phenomena, which are intermittent in nature.

The Cauchy distribution is characterized by the mean value \( c \) and by the half-width \( w \) in the following way:

\[ f(v) = \frac{w}{\pi (v-c)^2 + w^2}, \quad F(v) = \frac{1}{2} + \frac{1}{\pi} \arctan \left( \frac{v-c}{w} \right). \]  

(12.36)

The random distribution of the modulus of flow velocity in presence of coherent structures approaches the Cauchy distribution.

The graphs of the distribution functions described above can be seen in Figure 12.1.
11.4.5. Statistical moments

Other tools for description and analysis of random signals are statistical moments. They can be expressed by means of probability density. Let \( u \) be a random scalar variable and let \( v \) be its particular random quantity; then the mean value \( \bar{u} \) of the random variable \( u \) is

\[
\bar{u} = \int_{-\infty}^{\infty} v f(u) \, dv.
\]  

(12.37)

The similar relationship holds for mean value of an arbitrary function of the random variable \( Q(u) \)

\[
\bar{Q(u)} = \int_{-\infty}^{\infty} Q(v) f(v) \, dv.
\]  

(12.38)

In practical situations we often work with fluctuations of a random variable \( u' \) defined in the following way

\[
u' = u - \bar{u}.
\]  

(12.39)

Using fluctuations, central statistical moments of \( n^{th} \) order are defined:

\[
\mu_n \equiv u^n = \int_{-\infty}^{\infty} (v - \bar{u})^n f(v) \, dv.
\]  

(12.40)

For orders 0 and 1 there is \( \mu_0 = 1 \) and \( \mu_1 = 0 \). The central statistical moment of the second order represents the variance of the random variable

\[
\sigma^2 = \mu_2 = \int_{-\infty}^{\infty} (v - \bar{u})^2 f(v) \, dv.
\]  

(12.41)

The variability of a random variable is very often expressed in a form of a square root of its variance; this quantity has the same physical dimension as the variable itself and is called standard deviation or root mean square – r.m.s.

\[
std(u) = \sqrt{\mu_2} = \sqrt{\sigma^2}.
\]  

(12.42)

The principal characteristic of the fluctuation component of the velocity of flow in a given point is the intensity of fluctuations \( I_u \) or the intensity of turbulence \( T_u \). The general definition is

\[
I_u = T_u = \frac{\sigma^2}{\left| \bar{u} \right|^2},
\]  

(12.43)

and its value is often expressed in percentage points. Note that in the case of an isotropic phenomenon there is \( u'^2 = u'^3 = \bar{u}'^2 = u'^2 \) and then the expression \( T_u = \sqrt{\sigma^2 / \left| \bar{u} \right|^2} \) holds.

Evaluation of central statistical moments of the 3\(^{rd}\) and 4\(^{th}\) order is a common affair in turbulence theory. They also characterize distribution of random variable in a useful way. For the sake of practicability these moments are normalized, i.e. divided by, the respective power of variance in order to obtain a dimensionless number. The items thus obtained are skewness factor \( S \) and flatness factor \( F \) :

\[
S = \frac{\mu_3}{\mu_2^{3/2}} \quad \text{and} \quad F = \frac{\mu_4}{\mu_2^2}.
\]  

(12.44)

These factors are used for characterizing the random process and its fast and comfortable comparison with the normal Gauss distribution. For Gauss distribution there is \( S = 0 \) and \( F = 3 \).
11.4.6. Correlation function

For a stationary random process the autocorrelation function is defined as
\[ R(\tau) = u(t)u(t+\tau), \]
where \( u(t) \) is the signal under test and \( \tau \) is a shift in time.

The autocorrelation function has the physical dimension of the square of the random variable, which may be rather inconvenient in many cases. Therefore, the autocorrelation factor is calculated, which is dimensionless and it assumes values in the interval \( 0,1 \). Definition of the autocorrelation factor is
\[ \rho(\tau) = \frac{\overline{u(t)u(t+\tau)}}{u^2(t)}. \]

Another useful definition is the mutual correlation function \( R_{uv} \) and mutual correlation factor \( \rho_{uv} \) of the two random variables having their mean values equal to zero:
\[ R_{uv} = u(t)v(t), \quad \rho_{uv} = \frac{\overline{u(t)v(t)}}{\sqrt{\overline{u^2(t)}\overline{v^2(t)}}}. \]

The value of the mutual correlation factor indicates in a quantitative way the linear dependence of the two signals under test. A circumstance under which its value approaches 0 represents the necessary but not the sufficient condition for statistical independence of the two signals. On the other hand, the values 1 or -1 are a sufficient condition for linear dependence of the two signals.

11.4.7. Spectra

The definition of the velocity auto-covariance in time domain is
\[ R(s) = u'(t)u'(t+\tau), \]
where \( \tau \) represents a certain time shift.

Autocorrelation function is then defined as the normalized autocorrelation
\[ \rho(\tau) = R(\tau)/u^2(t). \]

The following relationships hold for autocorrelation function: \( |\rho(\tau)| \leq 1 \) and \( \rho(0) = 1 \); another fact is that the function \( \rho(\tau) \) is an even function, i.e. \( \rho(\tau) = \rho(-\tau) \) holds.

In case the process is stationary the autocorrelation and autocorrelation function are time-independent.

On the basis of autocorrelation some more important notions can be defined. Turbulent processes are of stochastic nature and the absolute value of their autocorrelation function is diminishing in time. Hence it follows that the integral of autocorrelation function converges as \( \tau \) approaches infinity. Therefore we can define the integral timescale as
\[ L = \int_0^\infty \rho(\tau)d\tau. \]

Another notion to be defined using to this end the Fourier transform is the frequency spectrum:
\[ E(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau = \frac{2}{\pi} \int_0^\infty R(\tau)\cos(\omega\tau)d\tau. \]

The second term follows from the circumstance that the \( R(\tau) \) is the even function. The inverse relationship the following also holds:
\[ R(\tau) = \int_{-\infty}^{\infty} E(\omega)e^{i\omega\tau}d\omega = 2\int_0^\infty E(\omega)\cos(\omega\tau)d\omega. \]
The quantities $R(\tau)$ and $E(\omega)$ contain equivalent information. The constants in front of the integration symbols in equations (12.51) and (12.52) can differ, the equations presented here are just the definition relationships.

The principal feature of $E(\omega)$ is that the integral $\int_{a_1}^{a_2} E(\omega) d\omega$ represents a contribution of frequencies within the range $a_1 \leq \omega < a_2$ to the value of variance $u^2(t)$. Another useful relationship is a possibility expressing the variance as the integral: $u^2(t) = \int_{-\infty}^{\infty} E(\omega) d\omega$.

By means of Wiener-Khinchin theorem the auto spectrum can be defined. The bilateral spectral density is defined as

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-2\pi if\tau} d\tau, \quad -\infty < f < \infty. \tag{12.53}$$

The inverse relationship also holds:

$$R(\tau) = \int_{-\infty}^{\infty} S(f) e^{2\pi if\tau} df. \tag{12.54}$$

For a routine use, the unilateral spectrum is more convenient; such spectrum is defined only for positive frequencies:

$$G(f) = 2S(f), \quad 0 \leq f < \infty. \tag{12.55}$$

The spectrum might be related to just a single signal and in this case we are talking about the auto spectrum; alternatively, if two signals are involved, we talk about mutual spectrum. Similarly the correlation function can assume either the autocorrelation or mutual correlation forms.

The Fourier transform has been used for many years by scientists and engineers for analysis and quantification of signals and the use of the Fourier transform became an indispensable tool of science and engineering a long time before computers were invented. Despite the fact that computers significantly enhanced the efficiency of work and widened the field of Fourier transform application, it remains one of the most demanding tasks of signal analysis due to the enormous volume of computations it requires.

The Fourier transform process is, according to its definition, continuous transformation, and the practical computations are performed in computers using numeric methods and therefore the practical outputs in this field are referred to as the Digital Fourier Transform – DFT.

At the beginning of the 1960s a new method was introduced, which made the efficiency of computations substantially higher and reduced the number of necessary computations of the Fourier transform evaluation. This method acquired the name Fast Fourier Transform – FFT; as the authors of this method are traditionally recognized as J.W. Cooley and J.W. Tukey. However, at a closer examination of the issue it turns out that the authorship of the mathematical procedure must be ascribed to Gauss who in 1805 de facto brought Fourier series into the body of mathematical knowledge. This new procedure can reduce the demand for computational power of the classical form of DFT, where the number of operations increases as $N^2$ ($N$ is the length of signal), to the reduced number $N \cdot \log N$ in the case of FFT. This means that, as an example, for the signal 1024 points long the FFT method is more than two orders of magnitude faster than the classical DFT.

We shall not deal in detail with the implementation of numeric calculation algorithms here. It is sufficient to say that using such algorithms the required results (defined by the Fourier transform) can be obtained by somewhat different and much more effective procedures rather than attempting to obtain these results directly from the definition relationships. These algorithms are extremely reliable and they are implemented in all respectable signal analysis software packages (e.g. MATLAB, LabVIEW, etc.).

The Fourier analysis is a classical method that can be applied only to the stationary signals, if the interpretation of the obtained results has to be meaningful. Therefore, in the cases of non-stationary signal, we must select an integration interval sufficiently short to allow the simplifying assumption that during this interval the signal can be considered statistically stationary.
However, for the analysis the comparatively long record of the signal is required. Another
disadvantage is that the method is sensitive to the wide-band noise present in the signal.

The Fourier transform represents an example of the integral transform determined by the
convolution product of the signal and the set of harmonic analysis functions.

\[ X(f,T) = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{i2\pi ft} dt \quad \text{for} \quad -\infty < f < \infty \]  \hspace{1cm} (12.56)

Here the expression \( X(f,T) \) represents Fourier transform of the signal \( x(t) \) having the
length \( T \).

At the present time the Fourier transform is a standard method with robust and effective
algorithms of implementation. Therefore there is a persistent trend towards a maximum use of
this tool; for example the problem at hand is being solved in spectral domain using Fourier
Figures.

The periodogram \( P(f,T) \) originates from application of the Fourier transform to the signal
of finite length \( T \):

\[ P(f,T) = \frac{1}{T} |X(f,T)|^2. \]  \hspace{1cm} (12.57)

The periodogram approaches the spectral power density or, in other words, to the spectrum of the
signal \( x(t) \):

\[ S(f) = \lim_{T \to \infty} P(f,T). \]  \hspace{1cm} (12.58)

It can be shown that for a signal of finite length \( T \) the estimated spectral power density
deviates and is inconsistent, i.e. the variance of its values does not diminish with the growing
length of the signal. The individual periodogram is therefore of no use for signal analysis due to
its excessive variance. This feature is particularly visible in case of digital form of the Fourier
transform where the band width and therefore also the frequency step is given by reciprocal of the
length of the signal; it is obvious that during calculation of the individual values they are not
centered and thus we obtain the values lopsided by a random error. Figure 12.2 shows the
difference between the periodogram and the spectrum.

![Figure 12.2 – Periodogram and spectrum of the random signal](image)

The acceptable estimate of the spectrum can be obtained by applying the method of
averaging. Averaging can be performed in two ways, but the result is the same in both cases. The
two ways in question are the averaging of periodograms and averaging of frequency bands.

When averaging a periodogram, the original signal of the length \( T \) is divided into \( p \) blocks
of the length \( T_p = T/p \). This brings about the change of the bandwidth from \( 1/T \) to \( 1/T_p \), which
is p-times greater. The spectrum estimate \( \tilde{S}(f) \) will be then

\[ \tilde{S}(f) = \frac{1}{T \cdot p} \sum_{k=1}^{p} |X(f,T)|^2. \]  \hspace{1cm} (12.59)

Therefore, in the case of discrete implementation of calculations we perform averaging of
values assigned to each separate frequency.
When averaging the frequency bands we take as the point of departure just one periodogram, which is characterized by its sufficiently narrow band. The number of bands is reduced during the procedure because a certain number of neighboring bands are averaged out. Another calculation option is to apply the sliding average method to the periodogram; thus the periodogram becomes smoother. This operation is equivalent to the application of the low-pass filter, which filters off high frequencies and brings about a de facto reduction of active bands.

Both manners of averaging should yield similar results. However, taking into consideration that the computational demands of the fast Fourier Transform are proportional to the product $N \cdot \log N$, it becomes obvious that the first method of averaging periodograms is the more effective of the two (when using the discrete Fourier transform, the difference becomes even more conspicuous).

Splitting the signal into blocks and applying a statistical treatment of each block separately is tantamount to application of the high-pass filter whereby the influence of the changes of the signal with the period longer that the length of one block is eliminated.

In practical cases the most often required information is the determination of the spectral power density from the recorded time series of measured signals. From these records the spectra evaluation can be obtained both in time and spatial domains using to this end Taylor hypothesis.

### 11.4.8. Wavelet transform

Wavelet transform represents a powerful tool of a joint signal analysis in time and frequency domains. For this purpose the so-called continuous variant of the wavelet transform is of particular importance.

There is also the “discrete” variant of the wavelet transform. It is used for the purpose of signal compression and it is also suitable for determining the “coherent” signal contents. The transform must be applied to that part of a signal that is related to the presence of vortex structures in the flow field, because if a suitable selection of mother wavelet is made the influence of these structures can be easily identified.

The result of the analysis is a two-dimensional graph that can be interpreted as the temporal development of instantaneous spectra. Wavelet transform is one of the most efficient variants of the joint signal analysis in time and frequency domains.

We shall present here the theory of wavelet transform for one-dimensional space characterized by the variable $x$, representing for example time. This theory can be easily extended into a space of an arbitrary number of dimensions using to this end the operation or rotation applied to the mother wavelet in addition to the operations of shift and dilatation.

“Mother wavelet” may be the real or complex function, which, to be “admissible” must meet the condition

$$ C_\psi = \int_0^\infty |\hat{\psi}(\kappa)|^2 \frac{d\kappa}{|\kappa|} < \infty, \quad (12.60) $$

where

$$ \hat{\psi}(\kappa) = \int_{-\infty}^{\infty} \psi(x) e^{-i2\pi\kappa x} dx \quad (12.61) $$

represents Fourier Figure of the mother wavelet and $\kappa$ is the wave number. Meeting the inequality (12.60) is the condition for existence of the reproduction core characterized by a finite energy and thus also for the existence of the inverse wavelet transform.

Therefore, the function $\psi$ is admissible as long as its mean value equals zero:

$$ \int_{-\infty}^{\infty} \psi(x) dx = 0 \text{ or } \hat{\psi}(\kappa = 0) = 0. \quad (12.62) $$

For practical reasons it is required that the mother wavelet is well localized both in the physical space of $x$ variable and in the Fourier space of variables $\kappa$. This means that the function $\psi$ must manifest fast damping in connection with growing value of $|x|$ and it must be smooth. It is also convenient to require that the higher moments are equal to zero:

---

13 Also “explanatory functions”
\[
\int_{-\infty}^{\infty} x^m \psi(x) \, dx = 0 \quad \text{for } m = 1, \ldots, M .
\]

Meeting of this condition guarantees that the single terms up to the order \( M \) are reproduced exactly. In the Fourier space this condition is equivalent to the requirement that Fourier Figure \( \tilde{\psi} \) continuously approaches zero when \( \kappa \) approaches zero:

\[
\left. \frac{d^m}{d\kappa^m} \tilde{\psi}(\kappa) \right|_{\kappa=0} = 0 \quad \text{for } m = 0, \ldots, M .
\]

By means of a mother wavelet \( \psi \) the family of wavelets can be generated that are characterized by translation and dilatation

\[
\psi_{l,x}(x') = \frac{1}{\sqrt{l}} \psi\left(\frac{x' - x}{l}\right),
\]

where \( l \) is the dilatation (or contraction) factor, \( l > 0 \) and \( x \) is the parameter of translation \( x \in \mathbb{R} \); the individual wavelets are normalized using the norm \( L^2 \). In the Fourier space we get the relationship

\[
\tilde{\psi}_{l,x}(\kappa) = \sqrt{l} \tilde{\psi}(l\kappa) e^{-i2\pi x}. \tag{12.66}
\]
equivalent to the equation (12.65).

Here the contraction \( 1/l \) corresponds to the dilatation \( l \) and the translation \( x \) corresponds to the rotation in a complex plane.

Selection of the mother wavelet has a decisive influence on the properties of the transform, in particular on its ability to distinguish details in time and spectral region.

In general, the wavelets can be divided according to different aspects. They can be purely real functions or the complex ones. By means of purely real wavelets the position of those parts of the signal can be localized that are “similar” to the wavelet. In periodic functions this identification takes place without any reference to the phase conditions. For detecting the amplitude the complex wavelets are far more convenient because they make possible an independent evaluation of both the amplitude and the phase, while in the real wavelets these data are merged. The examples of real and complex wavelets suitable for the analysis by continuous wavelet transform are presented in Figure 12.3; in the left column the wavelet is depicted in the physical space and in the right column in the Fourier space.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{wavelets.png}
\caption{Examples of mother wavelets}
\end{figure}
Morlet wavelet in Figure 12.3a is a harmonic function modulated by the Gauss function and its shapes in time and frequency domains are

\[
\psi(x) = \frac{1}{\pi^{1/4}} e^{i\kappa_0 x} e^{-x^2/2}, \quad \tilde{\psi}(\kappa) = \frac{1}{\pi^{1/4}} \theta(\kappa) e^{-(\kappa-\kappa_0)^2/2},
\]

(12.67)

where \(\theta(\kappa)\) is the Heaviside function. The full line represents the real part of the function and the dotted line represents the imaginary part thereof. This wavelet does not exactly meet the “admissibility” condition (12.62). The order \(\kappa_0\) determines the localization in time and frequency region. The higher the value of \(\kappa_0\), the better is the localization in the frequency region and the worse in the physical region; simultaneously it also becomes poorer in meeting the condition (12.62). In Figure 12.3a there is the Morlet wavelet for \(\kappa_0 = 6\). This wavelet manifests relatively good localization in the Fourier space but a worse one in the physical space.

By far better localization in the physical space (and therefore the worse one in the Fourier space) is manifested by the Paul wavelet\(^{14}\) in Figure 12.3b

\[
\psi(x) = \frac{2^m m!}{\sqrt{\pi} (2m)!} i^m \left(1-ix\right)^m, \quad \tilde{\psi}(\kappa) = \frac{2^m}{\sqrt{m(2m-1)!}} \theta(\kappa) \kappa^m e^{-\kappa}.
\]

(12.68)

Of the purely real wavelets we are presenting the \(m^{th}\) derivative of the gauss function (DOG) here:

\[
\psi(x) = (-1)^m \frac{d^m}{dx^m} \left[e^{-1/2} \right], \quad \tilde{\psi}(\kappa) = m(ik)^m e^{-\kappa^2/2}.
\]

(12.69)

In Figure 12.3c the second derivative of the Gauss function is presented (this often used Marr wavelet is also known as the Mexican hat).

The wavelets can, following the application of specific dilatation and translation, generate the complete base of orthogonal functions. This feature can be used to advantage and thereby to process practically any function without information loss. The representation in the space of wavelet factors is usually much more effective than that in the physical space and thus it is possible to use the wavelet method as the method of compression.

\[\text{Figure 12.4 – Example of the wavelet analysis of the intermittent signal}\]

\(^{14}\) Sometimes referred to as the Cauchy wavelet.
Figure 12.4 presents the example of wavelet analysis of the intermittent signal dependent on time \( t \), which can be seen in the upper part of the picture while in the lower part of the picture there is the wavelet transform where \( \tau \) is the wavelength of the wavelet (in logarithmic scale). In this example the Marr mother wavelet was used.

### 11.4.9. Proper orthogonal decomposition

The principle of the proper orthogonal decomposition method – POD was published by J.L. Lumley in 1967. The practical application of this method has become feasible only recently after accumulation of relevant data and acquisition of powerful computational tools. The method is based on the projection of a suitable coherent structure onto an experimentally determined velocity field. The dominant coherent structure maximizes this projection in terms of the minimal sum of squared deviations. The variation task of seeking the maximum is translated into the task of seeking solution of the Fredholm integral equation of the first type. The task is solved by solving the problem of proper functions and proper values (eigenfunctions and eigenvalues)

\[
\sum_{j=1}^{n_c} \int_{D} \Psi_\theta (y, y'; f) \hat{\Phi}^{(n)}(y'; f) dy' = \lambda^{(n)}(f) \hat{\Phi}^{(n)}(y; f),
\]

where \( n_c \) is the number of velocity components considered. The symmetric core is the correlation matrix \( \Psi_\theta (y, y'; f) \), given by the mutual spectrum of velocity components \( u_i, u_j \) and which corresponds to Fourier transform of the time-space correlation

\[
\Psi_\theta (y, y'; f) = \int_{-\infty}^{\infty} u_i(y, t) u_j(y', t + \tau) \exp(-2i\pi f \tau) d\tau.
\]

It can be shown that there is a countable number of eigenvalues and eigenfunctions of the problem. These are mutually orthogonal and create the complete basis, which can be used for complete reconstruction of the random velocity field. The core can be expanded into absolutely and uniformly convergent series of eigenfunctions; the turbulent kinetic energy is then the sum of contributions of the individual eigenfunctions.

In practical applications of this method it becomes apparent that for a sufficiently telling description of the velocity field it is sufficient to consider just several eigenfunctions of the lowest order.

The base found by means of the POD method can be conveniently used for converting the system to its discrete representation. Even using just a comparatively small number of these functions the system and its dynamical behavior can be obtained. Thus, we have a system with a low number of dimensions on our hands.

![Figure 12.5 – Cumulative energy of eigenfunctions](image)

In a way of example we are presenting here the results of experimental flow in the vicinity of the so called synthesized jet using the PIV (Particle Image Velocimetry) method. For the evaluation using the POD method about 5,000 vector maps of the velocity field were used. Figure 12.5 illustrates the cumulative energy contained in the individual eigenfunctions (mode number – eigenfunction on the horizontal coordinate is set in logarithmic scale). The fast convergence in a sense of energy is obvious: the first eigenfunction contains about 73 \% of
energy and the first 5 eigenfunctions contain more than 80% of the total energy. Figure 12.6 presents vorticity distributions of the first 4 eigenfunctions with the largest energy content. The jet is created by a generator with the estuary at the origin of coordinated on the wall and air is blown from left to right. We can clearly see the systems of vortex structures at different stages of development.

![Vorticity Distributions](image)

**Figure 12.6 – Eigenfunctions (proper functions)**

### 11.5. Similarity laws

In the study of mechanics of fluids the laws of similarity are used quite often.

The key dimensionless criterion related to turbulence is the Reynolds number \( \text{Re} = UL/\nu \), which characterizes a flow of viscous fluid. Let us see now, what in fact makes the Reynolds number such an important parameter. The velocity gradients \( \partial u/\partial x \) are of the \( U/L \) order and therefore the velocity components \( u \) manifest the change of \( U \) order at the distances of the order \( L \). These derivatives are typically subject to value changes of the order \( U/L \) at the distance \( L \) and the second derivatives \( \partial^2 u/\partial x^2 \) are therefore of the order \( U/L^2 \). These considerations offer the way to estimate the order of magnitude of the absolute terms of the N-S equation (4.20), which can be rewritten in the following manner:

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u. \tag{12.72}
\]

By the procedure described above we can estimate the size of some terms:

- **Inertial term:**
  \[ |(u \cdot \nabla)u| = \mathcal{O}\left(\frac{U^3}{L}\right). \tag{12.73} \]

- **Viscous term:**
  \[ |\nu \nabla^2 u| = \mathcal{O}\left(\nu U/L^2\right). \tag{12.74} \]

The function \( \mathcal{O}(\cdot) \) represents the estimate of the expression inside the brackets. From all these relationships we directly obtain

\[
\frac{|\text{inertial term}|}{|\text{viscous term}|} = \mathcal{O}\left(\frac{U^2/L}{\nu U/L^2}\right) = \mathcal{O}(\text{Re}). \tag{12.75}
\]

It can be seen that the Reynolds number describes the approximate ratio of inertial and viscous forces present in the stream of a real flow fluid. At the same time, this value offers some idea on the magnitude of the two key terms of the N-S equation.
Let us first consider the situation that the Reynolds number is large, i.e. \( \text{Re} \gg 1 \). We can gather from the relationship (12.75) that the viscous forces are from the global point of view negligible. Indeed, in a flow of fluid around a body at high Reynolds number the viscous effects are in most regions of the flow negligible and the only exception is a thin boundary layer where large velocity gradients can be found. The thickness of the boundary layer \( \delta \) depends on the Reynolds number in the following way:

\[
\frac{\delta}{L} = \mathcal{O}\left(\text{Re}^{-1/2}\right).
\]  

(12.76)

The high value of the Reynolds number is the necessary condition for validity of application of inviscid methods such as Euler equations. However, it is not a sufficient condition because in the case of stability loss and emergence of turbulence the inviscid model cannot be applied any longer.

Figure 12.7 presents comparison of two events differing in their respective Reynolds numbers. The first event is a droplet impinging on a flat surface while the second event is an explosion of a nuclear bomb. It can be seen that the global manifestations are rather similar to each other, but the structures are absolutely different.

![Figure 12.7 – Comparing \( \text{Re} = 10^2 \) (droplet) and \( \text{Re} = 10^9 \) (Nevada nuclear experiment)](image)
A review of some similarity numbers used in the theory of fluid dynamics

<table>
<thead>
<tr>
<th>Name</th>
<th>Formula</th>
<th>Physical contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler number</td>
<td>$\text{Eu} = \Delta p/\left(\rho \cdot U^2\right)$</td>
<td>Pressure difference related to kinetic energy</td>
</tr>
<tr>
<td></td>
<td>$\Delta p$ pressure difference</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\rho$ fluid density</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$U$ characteristic velocity</td>
<td></td>
</tr>
<tr>
<td>Grashof number</td>
<td>$\text{Gr} = g \cdot \beta \cdot \Delta T \cdot l^3 / \nu^2$</td>
<td>Ratio of buoyancy and viscous forces</td>
</tr>
<tr>
<td></td>
<td>$g$ gravity acceleration</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\beta$ volume factor of thermal dilatation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Delta T$ temperature difference</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$l$ characteristic length</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\nu$ kinematic viscosity</td>
<td></td>
</tr>
<tr>
<td>Knudsen number</td>
<td>$\text{Kn} = \lambda / l$</td>
<td>Flow of diluted gases</td>
</tr>
<tr>
<td></td>
<td>$\lambda$ molecular free path</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$l$ characteristic length</td>
<td></td>
</tr>
<tr>
<td>Mach number</td>
<td>$\text{Ma} = U/c$</td>
<td>Influence of fluid compressibility</td>
</tr>
<tr>
<td></td>
<td>$U$ characteristic velocity</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c$ velocity of sound</td>
<td></td>
</tr>
<tr>
<td>Nusselt number</td>
<td>$\text{Nu} = \left(h \cdot l\right) / \lambda$</td>
<td>Heat transfer between the body and the fluid</td>
</tr>
<tr>
<td></td>
<td>$h$ factor of heat transfer in forced convection</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$l$ characteristic length</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda$ thermal conductivity</td>
<td></td>
</tr>
<tr>
<td>Prandtl number</td>
<td>$\text{Pr} = \nu / \alpha$</td>
<td>Link between the velocity and temperature fields</td>
</tr>
<tr>
<td></td>
<td>$\nu$ kinematic viscosity</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\alpha$ thermal diffusivity</td>
<td></td>
</tr>
<tr>
<td>Rayleigh number</td>
<td>$\text{Ra} = \text{Gr} \cdot \text{Pr}$</td>
<td>Naturel convection</td>
</tr>
<tr>
<td>Reynolds number</td>
<td>$\text{Re} = (U \cdot l) / \nu$</td>
<td>Ratio of inertial and viscous forces</td>
</tr>
<tr>
<td></td>
<td>$U$ characteristic velocity</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$l$ characteristic length</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\nu$ kinematic viscosity</td>
<td></td>
</tr>
<tr>
<td>Strouhal number</td>
<td>$\text{St} = (f \cdot d) / U$</td>
<td>Similarity of periodic phenomena in flow fluid</td>
</tr>
<tr>
<td></td>
<td>$f$ fervency</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$d$ characteristic dimension</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$U$ characteristic velocity</td>
<td></td>
</tr>
</tbody>
</table>
11.6. A brief history of turbulence research

The brief listing of some historical milestones in the field of turbulence research is presented in the table below.

<table>
<thead>
<tr>
<th>Year</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>6th Century B.C.</td>
<td>Hérakleitos of Ephesus: “panta rhei” (all flows), “You never enter the same river” – reference to the turbulent nature of the world.</td>
</tr>
<tr>
<td>1st Century B.C.</td>
<td>Epicureanism, first atomists, Titus Lucretius Carus (Roman poet) the author of “De rerum natura” (On Nature) – didactic-epic poem. The turbulence is presented here as a cause of origin of all things.</td>
</tr>
<tr>
<td>15th Century A.D.</td>
<td>Leonardo da Vinci identified the two different states of a flow fluid and he introduces the notion of “la turbulenza”.</td>
</tr>
<tr>
<td>1687</td>
<td>I. Newton formulates the principle laws of mechanics.</td>
</tr>
<tr>
<td>1739</td>
<td>D. Bernoulli publishes the “Bernoulli equation”.</td>
</tr>
<tr>
<td>1822</td>
<td>C.L. Navier derived the moment equations describing the behavior of viscous fluid.</td>
</tr>
<tr>
<td>1839</td>
<td>G.H.L. Hagen repeated the “discovery” of the two states of a fluid flow through a pipe.</td>
</tr>
<tr>
<td>1871</td>
<td>Lord Kelvin (W. Thomson) studies the stability of laminar flow.</td>
</tr>
<tr>
<td>1877</td>
<td>J. Boussinesq introduces the idea of turbulent viscosity.</td>
</tr>
<tr>
<td>1883</td>
<td>O. Reynolds carries out his experiments investigating transition of laminar flow through a pipe to the turbulent one. He identifies the Reynolds number.</td>
</tr>
<tr>
<td>1887</td>
<td>Lord Kelvin introduces the notion of “turbulence”.</td>
</tr>
<tr>
<td>1895</td>
<td>Reynolds decomposition.</td>
</tr>
<tr>
<td>1904</td>
<td>L. Prandtl introduces the notion of “boundary layer”.</td>
</tr>
<tr>
<td>1907</td>
<td>W. Orr formulates the equation of laminar flow stability.</td>
</tr>
<tr>
<td>1909</td>
<td>D. Riabuchinsky invented the method of turbulent flow measurement using the hot wire (constant current mode).</td>
</tr>
<tr>
<td>1912</td>
<td>J.T. Morris invented the anemometry method with constant temperature of the hot wire; this method makes possible measurement of velocity fluctuations of very high frequencies.</td>
</tr>
<tr>
<td>1914</td>
<td>A. Einstein proposed the use of covariance and mutual correlations for the study of signals with fluctuations.</td>
</tr>
<tr>
<td>1914</td>
<td>E. Buckingham formulates his π-theorem, the theoretical foundation of the dimensional analysis.</td>
</tr>
<tr>
<td>1921, 1935</td>
<td>G.I. Taylor used statistical method for processing the turbulent signals.</td>
</tr>
<tr>
<td>1922</td>
<td>L.F. Richardson discovered hierarchy of vortices (cascade) in turbulent flow.</td>
</tr>
<tr>
<td>1925</td>
<td>L. Prandtl introduces the notion of “mixing length” as the characteristics of velocity fluctuations.</td>
</tr>
<tr>
<td>1930</td>
<td>T. von Kármán formulates “the wall law” in turbulent boundary layer.</td>
</tr>
<tr>
<td>1938</td>
<td>G.I. Taylor discovered the mechanism of vorticity generation by stretching of vortices.</td>
</tr>
<tr>
<td>1941</td>
<td>A.N. Kolmogorov formulates the theory of homogeneous isotropic turbulence K41.</td>
</tr>
<tr>
<td>1942-6</td>
<td>J. von Neumann used for the first time a computer to tackle a problem of fluid mechanics (within the framework of the Manhattan project).</td>
</tr>
<tr>
<td>1943</td>
<td>S. Corsin discovered a sharp boundary between laminar and turbulent regions.</td>
</tr>
<tr>
<td>Year</td>
<td>Event</td>
</tr>
<tr>
<td>------</td>
<td>-------</td>
</tr>
<tr>
<td>1949</td>
<td>G. Batchelor and A. Townsend discovered small-scale intermittency phenomena in the developed turbulent flow.</td>
</tr>
<tr>
<td>1951</td>
<td>H.W. Emmons describes the turbulent stains.</td>
</tr>
<tr>
<td>1952</td>
<td>E. Hopf formulates his variation equation.</td>
</tr>
<tr>
<td>1952</td>
<td>T. Theodorsen posits a hypothesis that the turbulent flow field is made of coherent structures.</td>
</tr>
<tr>
<td>1962</td>
<td>A.N. Kolmogorov carries out the correction of his K41 theory adopting consideration of small scale intermittency phenomena – K62.</td>
</tr>
<tr>
<td>1963</td>
<td>E. Lorenz introduces the notion of “deterministic chaos”.</td>
</tr>
<tr>
<td>1967</td>
<td>S.J. Kline describes mechanism of generating Reynolds tension via the “bursting phenomenon”.</td>
</tr>
<tr>
<td>1975</td>
<td>B. Mandelbrot introduces the notion of fractal.</td>
</tr>
<tr>
<td>1977</td>
<td>I. Prigogin worked out the theory of self-organization of large complex dynamical systems.</td>
</tr>
<tr>
<td>1995</td>
<td>U. Frisch introduces the notion of “multifractal” in relation to the structures of turbulent regions.</td>
</tr>
<tr>
<td>2000</td>
<td>Elucidating of principal properties of the Navier-Stokes equations was proclaimed as one of the mathematical problems of the 3rd millennium.</td>
</tr>
</tbody>
</table>