FINITE ELEMENT APPROXIMATION OF FLUID STRUCTURE INTERACTION USING TAYLOR-HOOD AND SCOTT-VOGELIUS ELEMENTS

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Abstract
This paper addresses the problem of fluid flow interacting a vibrating solid cylinder described by one degree of freedom system and with fixed airfoil. The problem is described by the incompressible Navier-Stokes equations written in the arbitrary Eulerian-Lagrangian (ALE) formulation. The ALE mapping is constructed with the use of a pseudo-elastic approach. The flow problem is numerically approximated by the finite element method (FEM). For discretization of the fluid flow, the results obtained by both the Taylor-Hood (TH) element and the Scott-Vogelius (SV) finite element are compared. The TH element satisfies the Babuška-Brezzi inf-sup condition, which guarantees the stability of the scheme. In the case of the SV element the mesh, that is created as a barycentric refinement of regular triangulation, is used to satisfy the Babuška-Brezzi condition. The numerical results for two benchmark problems are shown.

Keywords: finite element method, arbitrary Lagrangian-Eulerian method, Scott-Vogelius element, Taylor-Hood element.

1 Introduction
The numerical approximations of fluid-structure interaction are crucial in various scientific and engineering applications such as, e.g., problem of aeroelastic stability of aircraft wings, the air flow interacting with wind turbine blades, hydrodynamic compressors, etc. Although in various situations it is possible to consider a simplified model of incompressible fluid flow, it still involves addressing numerous numerical challenges such as handling the incompressibility constraint and nonlinear convective terms (e.g., [19, 18, 4]). Additionally, it requires incorporating the time change of the computational fluid domain, which is usually treated with the aid of the arbitrary Lagrangian-Eulerian (ALE) method, see, e.g. [19].

The primary objective of this paper is to compare numerical performance of finite element approximations of the Navier-Stokes equations. In this case, there exist various possibilities on how to choose a couple of finite elements (FE) (e.g., [11, 10]), but in this paper only two choices, Taylor-Hood (TH) and Scott-Vogelius (SV) elements satisfying the Babuška-Brezzi (BB) inf-sup condition, are considered.

The TH finite element is the standard well-known option with continuous piecewise quadratic velocities and continuous piecewise linear pressures. The drawback of this element is that the divergence constraint is satisfied only discretely. Consequently, the grad-div stabilization is required to be used for the high Reynolds numbers cases, see [7, 10]. In order to overcome this, the SV finite element is used, featuring continuous piecewise quadratic velocities and discontinuous piecewise linear pressures, see [10, 7]. To meet the BB condition, the finite element approximation space is constructed over a barycentric refinement of an admissible triangulation. This choice ensures a strong guarantee of the divergence constraint on each element, providing better stability, see [8].

The paper presents a comparison of the numerical results for both TH and SV finite elements obtained by an in-house solver written in the C programming language. The benchmark problems addressed involve non-stationary incompressible flow around a vibrating cylinder and around the fixed NACA 0012 airfoil, see [3, 15].

2 Governing equations
In this section, the problem of the interaction between the fluid flow and a rigid structure motion is mathematically described by the system of ordinary differential equations describing the rigid body
motion coupled with the incompressible Navier-Stokes equations within the arbitrary Eulerian-Lagrangian (ALE) formulation describing the fluid flow motion in a time-dependent domain.

2.1 Navier-Stokes equations

Let us consider a bounded computation domain \( \Omega_t \subset \mathbb{R}^2 \) with continuous Lipschitz boundary consisting of three disjoint parts \( \partial \Omega_t = \Gamma_D \cup \Gamma_O \cup \Gamma_W \). Furthermore, the domain \( \Omega_t \) is assumed to be polygonal and fully occupied by the fluid at any time \( t \in [0, T] \). The incompressible viscous flow in the domain \( \Omega_t \) is mathematically described by the incompressible Navier-Stokes equations in the ALE formulation. The ALE method is based on the ALE mapping \( A_t \), which transforms the reference domain configuration \( \Omega_0 \) onto the current domain \( \Omega_t \), i.e.

\[
A_t : \Omega_0 \rightarrow \Omega_t, \quad X \mapsto x(X, t) = A_t(X), \quad X \in \Omega_0, \ t \in (0, T).
\]

The ALE mapping is an extension of the mapping which maps the reference position of the interface \( \Gamma_{W_0} \) on \( \Gamma_W \), where \( \Gamma_{W_0} \) is the surface of the cylinder in the reference domain \( \Omega_0 \) and \( \Gamma_W \) is the surface of the cylinder at time \( t \), see, e.g. [19].

Further, the Navier-Stokes equations in the ALE formulation read: Find the velocity \( u(x, t) : \Omega_t \rightarrow \mathbb{R}^2 \) and the kinematic pressure \( p(x, t) : \Omega_t \rightarrow \mathbb{R} \) which satisfy

\[
\begin{align*}
\frac{D^A}{Dt} u + [(u - w) \cdot \nabla] u - \nu \Delta u + \nabla p &= 0 \quad \text{in } \Omega_t, t \in (0, T], \\
\nabla \cdot u &= 0 \quad \text{in } \Omega_t, t \in (0, T],
\end{align*}
\]

where \( \nu \) is the kinematic viscosity, \( \frac{D^A}{Dt} \) denotes the ALE derivative, and \( w = \partial A^t / \partial t \) is the domain velocity, see [19].

Eqs. (1) are equipped with the boundary conditions

\[
\begin{align*}
u(x, t) &= g(x, t) \quad \text{on } \Gamma_D \times (0, T], \\
u(x, t) &= w(x, t) \quad \text{on } \Gamma_W, \ t \in (0, T], \quad (2a) \\
-(p - p_{ref})\mathbf{n} + \nu \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_O \times (0, T], \quad (2b)
\end{align*}
\]

where \( \mathbf{n} \) denotes the unit outward normal vector to \( \partial \Omega_t \) and \( p_{ref} \) represents a reference pressure value at the outlet, see [9].

2.2 Vibration of the cylinder

In this paper, the rigid cylinder with one degree of freedom is assumed. Its motion is described by the ordinary differential equations for the displacement \( \ddot{Y} \) in nondimensional form

\[
\ddot{Y} + \left( \frac{4\pi \xi}{U_r} \right) \dot{Y} + \left( \frac{4\pi^2}{U_r^2} \right) Y = \frac{C_l}{2M^*}.
\]

Here, \( \ddot{Y} \) represents the vertical acceleration of the cylinder, \( \dot{Y} \) denotes its velocity, \( \xi \) stands for the structural damping ratio, \( U_r = \frac{v}{f_n \rho} \) is the reduced velocity of the cylinder (with \( f_n \) representing the natural frequency of the cylinder), \( M^* \) corresponds to the reduced mass of the rigid cylinder (\( M^* = \frac{M}{\rho A} \)), and \( C_l \) is the lift coefficient, see [3].

3 Discretization of the fluid flow problem

To approximate Problem (1), we start with semi-discretization of the time. Using a constant time step \( \Delta t > 0 \), the time interval \( (0, T) \) is equidistantly divided into time intervals \( (t_n, t_{n+1}) \) with \( t_n = n \Delta t \). The velocity is approximated at time step \( t_n \in (0, T] \) by

\[
u^n(x) \approx \nu(x, t_n) \quad \text{for } x \in \Omega_n,
\]
and the pressure is approximated as

\[ p^n(x) \approx p(x, t^n) \quad \text{for } x \in \Omega_{t^n}. \]

The domain velocity at time instant \( t_{n+1} \) is approximated by \( \mathbf{w}^{n+1}(x) = \mathbf{w}(x, t_{n+1}) \) for \( x \in \Omega_{t_{n+1}} \).

The ALE derivative is approximated at fixed time step \( t_{n+1} \) by the second-order two-step backward difference formula. As a result, the implicit scheme is obtained

\[
\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} + ((\mathbf{u}^{n+1} - \mathbf{w}^{n+1}) \cdot \nabla)\mathbf{u}^{n+1} - \nu \Delta \mathbf{u}^{n+1} + \nabla p^{n+1} = 0,
\]

\[
\nabla \cdot \mathbf{u}^{n+1} = 0,
\]

where \( \tilde{\mathbf{u}}^t \) is the transformed velocity \( \mathbf{u}^t \) from \( \Omega_t \) onto \( \Omega_{t_{n+1}} \), i.e. \( \tilde{\mathbf{u}}^t = \mathbf{u}^t \circ A_t \circ A_{t_{n+1}}^{-1} \). Problem (4) is closed by the boundary conditions \((2a-2c)\).

### 3.1 Spatial Discretization

We use the FEM for discretization of Problem (4). The starting point of the FEM is the weak formulation. First, we consider a fixed time instant \( t_{n+1} \), and provide a simplified notation: \( \mathbf{u} = \mathbf{u}^{n+1}, \mathbf{w} = \mathbf{w}^{n+1}, p = p^{n+1}, \) and \( \Omega = \Omega_{t_{n+1}} \).

Furthermore, the velocity test space \( \mathbf{V} \) and the pressure test space \( \mathbf{Q} \) are defined as

\[ \mathbf{V} = \{ \varphi \in H^1(\Omega) : \varphi(x) = 0 \ \forall x \in \Gamma_D \cup \Gamma_W \}, \quad \mathbf{Q} = L^2(\Omega), \]

where \( H^1(\Omega) = [H^1(\Omega)]^2 \) is the vector Sobolev space and \( L^2(\Omega) \) is the Lebesgue space, see [14].

Let us introduce the scalar product \( (\mathbf{u}, \mathbf{v})_\Omega = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, dx \) in \( L^2(\Omega) \) and the forms

\[
a(U^*, U, V) = \frac{3}{2\Delta t}(\mathbf{u}, \mathbf{v})_\Omega + \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega + c(\mathbf{u}^*, \mathbf{u}, \mathbf{v}) - (\mathbf{p}, \nabla \cdot \mathbf{v})_\Omega - (\nabla \cdot \mathbf{u}, \mathbf{q})_\Omega,
\]

\[
F(V) = \frac{1}{2\Delta t}(4\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^{n-1}, \mathbf{v})_\Omega,
\]

(4)

where \( U^* = (u^*, p), U = (u, p), \) and \( V = (v, q) \), for more detailed description see [11]. In this work, the term \( \mathbf{u}^* \) is linearized by \( \mathbf{u}^* = 2\bar{\mathbf{u}}^n - \mathbf{u}^{n-1} \), see [18].

Now, we take arbitrary test functions \( \mathbf{v} \in \mathbf{V} \) and \( q \in \mathbf{Q} \). By the former function multiply the first equation in (4), and by the latter function multiply the second equation in (4). In addition, integrate over the domain \( \Omega \), apply Green’s theorem to the pressure gradient \((\nabla \mathbf{p})\), and to the viscous term \( -\nu \Delta \mathbf{u} \). Finally, taking into account the boundary conditions \((2a-2c)\), the weak formulation is obtained in the form: Find \( U = (\mathbf{u}, p) \in \mathbf{V} \times \mathbf{Q} \) which satisfy

\[
a(U^*, U, V) = F(V), \quad \forall V \in \mathbf{V} \times \mathbf{Q},
\]

(5)

and \( \mathbf{u} \) the conditions \((2a-2b)\).

Further, the admissible triangulation \( \tau_h \) of the domain \( \Omega \) is defined (see [8]), and on this triangulation, the following finite element subspaces are used \( \mathbf{V}_h \subset \mathbf{V} \) as the velocity subspace and \( \mathbf{Q}_h \subset \mathbf{Q} \) as the pressure subspace. Generally, finite element subspaces consist of piecewise polynomial functions. In this paper, two choices of finite spaces are considered. First one the so-called Taylor-Hood element

\[
\mathbf{V}_h = \{ \varphi \in C(\overline{\Omega}) : (\varphi|_K \in P_2(K), \forall K \in \tau_h) \} \cap \mathbf{V},
\]

(6)

\[
\mathbf{Q}_h = \{ \varphi \in C(\overline{\Omega}) : (\varphi|_K \in P_1(K), \forall K \in \tau_h) \}.
\]

(7)

which satisfies the BB condition on the regular triangulation, but the divergence is satisfied only discretely, see [7]. To threaten this, the Scott-Vogelius element is used, which has the same space for the velocity (6), but the pressure space is

\[
\mathbf{Q}^{disc}_h = \{ \varphi : \overline{\Omega} \to \mathbb{R} : (\varphi|_K \in P_1(K), \forall K \in \tau_h) \}.
\]

The main difference is that this space is continuous only in each element and the mesh is constructed over the barycentric refined mesh, to secure the BB condition, see [7, 10, 6].
Finally, the discrete version of Problem (5) reads: Find $U_h \in \mathbf{V}_h \times Q_h$ which satisfy
\begin{equation}
 a(U_h^*, U_h, V_h) = f(V_h),
\end{equation}
for all $V_h \in \mathbf{V}_h \times Q_h$.

The discrete problem requires stabilization, as we mainly focus on flow with dominant convection, see [13]. Additionally, as the TH element satisfies the divergence constraint only discretely, it needs grad-div stabilization due to non-physical oscillations, see [12, 16]. It is the main advantage of the SV element over the TH that if one uses the SV element, only stabilization due to dominant convection is required. On the other hand, the classical Streamline Upwind Petrov-Galerkin (SUPG) method cannot be used due to the coupling of pressure $p$ and velocity $u$, see [20, 2]. Hence, the SUPG method is used only with the TH element, and with the SV element the classical streamline diffusion is employed in our numerical schemes.

4 Numerical simulations

In this section, the results of numerical simulations are discussed, such as the problem of the flow around the vibrating cylinder and flow around the NACA0012 airfoil.

4.1 Flow around the movable cylinder

The initial state of the domain, denoted as $\Omega_t$, is in Fig. 1. A cylinder is placed at the coordinates $[x, y] = [19, 20]$. The Dirichlet boundary condition is specified as $g = (1, 0)$ at the inlet $\Gamma_{D,1}$, and at the wall $\Gamma_{D,2}$, a zero-velocity condition is applied. On the surface of the cylinder, denoted as $\Gamma_W$, a Dirichlet boundary condition of the form $u = w$ is employed. The cylinder has one degree of freedom, which means that it is movable in vertical direction. Its position is obtained by solving Problem (3) using the 4-th order Runge-Kutta method and the fluid flow is modeled by solving Eq. (8).

Two meshes were performed for the computations, because the SV element requires a mesh which is created as barycenter refinement from regular mesh, and due to the discontinuity it leads to a much larger system. To investigate the two elements, meshes providing similar numbers of unknowns are used. For instance, the initial mesh (A) for the TH element entails solving a system with 89,519 unknowns, while the utilization of the second mesh (B) for the SV element yields a system with 90,798 unknowns. The flow problem around the cylinder is characterized by the Reynolds number $Re = \frac{U_\infty D}{\nu}$, where $U_\infty$ is the velocity of the free stream. This configuration ensures that the Reynolds number (Re) is fixed at 150, aligning with the reference data from [3].

Calculations were carried out for various scenarios that involved different natural frequencies of the cylinder. Throughout these computations, a zero damping ratio ($\xi = 0$) is consistently applied (to obtain the maximum amplitude of vibration) and the reduced mass ($M^* = 2$), to mimic the conditions outlined in [3].
A Von Karman vortex street is formed in the wake of the cylinder, consequently, by oscillations in aerodynamic forces the vibration of the cylinder arises. There is a comparison between the SV and the TH element in Fig 3, where there is the dependence of the maximum amplitude on the reduced velocity ($U_r$). If the natural frequency is close to the frequency of the vortices, there is a phenomenon called resonance. The frequency of the vortices is characterized by the Strouhal number $St = fD/U_{\infty}$, where $f$ is the vortex shedding frequency. For the fixed cylinder, it is $St \approx 0.185$, see [5]. However, the $St$ number is higher for a movable cylinder, e.g., for $U_r = 4$ the Strouhal number is $St = 0.22$. This might be due to the influence of the movement of the cylinder on the forming of the vortex shedding.

The observations are that the schemes with both elements are able to capture the resonance. The interval in which the resonance occurs agrees with the reference data [3], i.e., $U_r \in [4, 7]$. The highest amplitude is for $U_r = 4$ and then the amplitude decreases with increasing $U_r$.

4.2 Flow around the NACA0012 airfoil

The second problem is flow around the NACA0012 airfoil. The comparison of the lift coefficient ($C_l$) for different angles of attack $\alpha$ (AoA) with the reference data from [15] is given.

The airfoil has the chord length $c$ and the domain of the problem is shown in Fig. 2. The Dirichlet boundary condition is used at the $\Gamma_{D,1}$ inlet and at the surface of the airfoil $\Gamma_{D,2}$. Furthermore, the do-nothing condition is used at the outlet $\Gamma_O$. The problem is characterized by the Reynolds number $Re = \frac{U_{\infty}c}{\nu}$. Our simulations were performed for the $Re = 6,000,000$, same as in [15]. The two meshes were used. Initial mesh A for the TH element involves solving a system with 120,700 unknowns, whereas performing the second mesh B for the SV element results in a system with 238,290 unknowns. The numerical scheme with the SV element requires many more unknowns, as it is only the first order of accuracy due to stabilization.

In our simulations, the AoA was in the range $0^\circ - 10^\circ$. In Figs. 4 and 5, there are the magnitudes of the velocity fields for case $\alpha = 2^\circ$. The numerical scheme with the TH element captures the vortices better (Fig. 4) than the scheme with the SV element (Fig.5). This is more visible in Figs. 6 and 7, where is the case $\alpha = 10^\circ$. The structure of the flow includes more vortices, which are not captured properly with the SV element. This is due to the use of streamline diffusion in the scheme with the SV element. Therefore, the numerical scheme requires a finer mesh, since it is only 1st order of accuracy. Another option is to improve the stabilization technique, see [20, 2, 17].
In addition, a comparison with the experimental data of [15] is given in Fig. 8. As the AoA increases, so does the lift. It is observed that with the TH element, the mean lift is similar to the experimental data, while the SV element underestimates the lift coefficient for the $\alpha = 10^\circ$. This is because of the use of only streamline diffusion and the degradation of the method to the first order.

Figure 4: Magnitude of the velocity ($||u||_\infty$) for the $\alpha = 2^\circ$ with using the TH element.

Figure 5: Magnitude of the velocity ($||u||_\infty$) for the $\alpha = 2^\circ$ with using the SV element.

Figure 6: Magnitude of the velocity ($||u||_\infty$) for the $\alpha = 10^\circ$ with using the TH element.

Figure 7: Magnitude of the velocity ($||u||_\infty$) for the $\alpha = 10^\circ$ with using the SV element.

5 Conclusion

In this paper, the problem of numerical approximation of incompressible fluid flow problems was addressed. The performance of the finite element method was tested on two problems: first, the flow around the movable cylinder and second, the flow around the fixed NACA0012 airfoil. The problem was mathematically described by the incompressible Navier-Stokes system of equations in the arbitrary Lagrangian-Eulerian (ALE) formulation and coupled with the motion equation for the cylinder. The discretization using the Taylor-Hood (TH) $P_2/P_1$ element or the Scott-Vogelius (SV) $P_2/P_1^{disc}$ element was used and the numerical results were compared for the cross-flow movable cylinder benchmark, see [3]. The second test case was the flow around the NACA0012 airfoil, where the comparison with the experimental data of [15] was shown.

In the first case, the numerical results obtained by both methods align well with the reference data [3] regarding resonance occurrences within the specified range of the reduced velocity of the cylinder. In the second case, the lift coefficients obtained by both methods agree well up to the angle of attack 5 degrees with the TH element being slightly closer to the experimental data from [15]. For the angle of attack 10 degrees, the result obtained by the SV element differs significantly from the experimental data, whereas the result obtained by the use of stabilized TH finite element agrees well with the experimental data. This behaviour is probably caused by the
Figure 8: Comparison of the lift coefficients $C_l$ obtained with the TH element and SV element with the experimental results from [15].

use of different stabilization, where for the TH element the Streamline Upwind Petrov-Galerkin (SUPG) stabilization together with the grad-div stabilization is used, whereas for the SV element only the streamline diffusion method was applied because it is not possible to use the SUPG stabilization for the SV element due to the stronger coupling of the velocity and the pressure. On the other hand, the scheme with the SV element does not require grad-div stabilization as it already satisfies the divergence constraint strongly and the artificially added streamline diffusion can be replaced in the future by local projection stabilization, see [6].

In the first benchmark problem of flow around the movable cylinder results of both the SV element and the TH element are comparable with a similar number of unknowns even though the SV should provide theoretically better convergence rate. Overall, the results with the use of both elements are similar in the case where no stabilization is needed. The SV is computationally more expensive, but it has an important advantage that the divergence constraint is satisfied more precisely, which should result in a better approximation of the pressure, see [1]. However, this effect is suppressed in the presented results with the use of grad-div stabilization of the TH approximations, see [12, 16]. The performance of the approximations by the SV element is negatively influenced by the use of only streamline diffusion stabilization, which formally decreases the order of convergence. This problem can be overcome in future by using an advanced stabilization method such as a local projection method or variational multiscale method, see [6, 17].

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