AN APPROXIMATION METHOD FOR NON-STATIONARY STOKES FLOW

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Abstract
We consider a first order implicit time stepping procedure (Euler scheme) for the non-stationary Stokes equations in smoothly bounded domains of \( \mathbb{R}^3 \). Using energy estimates we can prove optimal convergence properties in the Sobolev spaces \( H^m(G) \) \((m = 0, 1, 2)\) uniformly in time, provided that the solution of the Stokes equations has a certain degree of regularity.

Keywords: Stokes Equations, Semi-Discretization, Euler Method, Energy Estimates

1 Introduction and Notation

Let \( T > 0 \) be given and \( G \subset \mathbb{R}^3 \) be a bounded domain with a sufficiently smooth compact boundary \( S \). In \((0, T) \times G\) we consider the non-stationary Stokes equations ([10],[11])

\[
\partial_t v - \nu \Delta v + \nabla p = F, \quad \text{div} \; v = 0, \quad v|_{\partial G} = 0, \quad v|_{t=0} = v_0. \tag{1}
\]

These equations describe the linearized motion of a viscous incompressible fluid [4]: The vector \( v = (v_1(t,x), v_2(t,x), v_3(t,x)) \) represents the velocity field, and the scalar \( p = p(t,x) \) the kinematic pressure function of the fluid at time \( t \in (0,T) \) and position \( x \in G \). The constant \( \nu > 0 \) is the kinematic viscosity, and the external force density \( F \) together with the initial velocity \( v_0 \) are the given data. The condition \( \text{div} \; v = 0 \) means the incompressibility of the fluid, and \( v = 0 \) on the boundary \( S \) expresses the no-slip condition, i.e. the fluid adheres to the boundary.

It is the aim of the present paper to develop an approximation method for the solution of (1) using semi-discretization in time ([3],[9]). Roughly speaking, the method transforms (1) into a finite number of certain boundary value problems for functions not depending on time [12]. Concretely, let us consider the following semi-discrete first order Euler approximation scheme for the Stokes equations (1): Setting

\[
h = T/N > 0, \quad t_k = k \, h \quad (k = 0, 1, \ldots, N),
\]

we approximate the solution \( v, p \) of (1) at time \( t_k \) by the solution \( v^k, p^k \) \((k = 1, 2, \ldots, N)\) of the following equations in \( G \):

\[
(v^k - v^{k-1}) h^{-1} - \nu \Delta v^k + \nabla p^k = h^{-1} \int_{(k-1)h}^{kh} F(t) \, dt, \tag{2}
\]

\[
\text{div} \; v^k = 0, \quad v^0 = v_0, \quad v^k|_{\partial G} = 0.
\]

Here \( F \) and \( v_0 \) are the given data. Thus for every \( k = 1, 2, 3, \ldots, N \) we have to determine in \( G \) the solution \( v^k, q^k \) of the Stokes resolvent boundary value problem [12]

\[
(\lambda - \Delta) v^k + \nabla q^k = F^{\lambda,k-1}, \quad \text{div} \; v^k = 0, \quad v^k|_{\partial G} = 0, \tag{3}
\]

with \( \lambda = (\nu h)^{-1} > 0, q^k = p^k/\nu \), and

\[
F^{\lambda,k-1}(x) = \lambda \left( v^{k-1}(x) + \int_{(k-1)h}^{kh} F(t,x) \, dt \right). \tag{4}
\]
This system can be solved numerically, for example, using methods of hydrodynamical potential theory combined with a boundary element method for the spatial discretization. These topics, however, are not addressed in this paper.

Let us now introduce our notations. Throughout the paper, \( G \subset \mathbb{R}^3 \) is a bounded domain having a compact boundary \( S \) of class \( C^2 \). In the following, all functions are real valued. As usual, \( C_0^\infty (G) \) denotes the space of smooth functions defined in \( G \) with compact support, and \( L^2(G) \) is the Hilbert space of square-integrable functions, equipped with scalar product and norm

\[
(f, g) = \int_G f(x)g(x)\,dx, \quad ||f|| = (f, f)^{\frac{1}{2}},
\]

respectively [2].

For functions \( f, g \in L^2(G) \) we need the following well-known relations [6]:

\[
\begin{align*}
(f - g, f + g) &= ||f||^2 - ||g||^2, \\
(f - g, 2f) &= ||f||^2 - ||g||^2 + ||f - g||^2, \\
2(f, g) &\leq 2||f|| ||g|| \leq ||f||^2 + ||g||^2.
\end{align*}
\]

(5)

The Sobolev space \( H^m(G) \) \((m = 0, 1, 2, \ldots)\) is the space of functions \( f \) such that \( D^\alpha f \in L^2(G) \) for all \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3 \) with \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq m \). Its norm is denoted by

\[
||f||_m = ||f||_{H^m(G)} = \left( \sum_{|\alpha| \leq m} ||D^\alpha f||^2 \right)^{\frac{1}{2}},
\]

where \( D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3} \) with \( D_k = \frac{d}{dx_k} \) \((k = 1, 2, 3)\) is the distributional derivative.

The completion of \( C_0^\infty (G) \) with respect to \( || \cdot ||_m \) is denoted by \( H^m_0(G) \) \((H^0_0(G) = H^0(G) = L^2(G))\). If \( f \in H^0_0(G) \), in particular, we have Poincaré’s inequality

\[
||f||^2 \leq C_G ||\nabla f||^2,
\]

(6)

where here the constant \( \lambda_1 = C_G^{-1} \) is the smallest eigenvalue of the Laplace operator \(-\Delta\) in \( G \) with zero boundary condition [8].

The spaces \( C_0^\infty (G)^3, L^2(G)^3, H^m(G)^3, \ldots \) are the corresponding spaces of vector fields \( u = (u_1, u_2, u_3) \).

Here norm and scalar product are denoted as in the scalar case, i. e. for example,

\[
(u, v) = \sum_{k=1}^3 (u_k, v_k), \quad ||u|| = (u, u)^{\frac{1}{2}} \quad = \int_G |u(x)|^2\,dx^{\frac{1}{2}},
\]

where \( |u(x)| = (u_1(x)^2 + u_2(x)^2 + u_3(x)^2)^{\frac{1}{2}} \) is the Euclidian norm of \( u(x) \in \mathbb{R}^3 \).

The completion of

\[
C_0^\infty_{0, \sigma}(G)^3 = \{ u \in C_0^\infty(G)^3 | \text{div } u = 0 \}
\]

with respect to the norm \( || \cdot || \) and \( || \cdot ||_1 \) are important spaces for the treatment of the Stokes equations ([4], [8], [10]). They are denoted by

\[
H(G)^3, \quad V(G)^3,
\]

respectively. In \( H^1_0(G)^3 \) and \( V(G)^3 \) we also use

\[
(\nabla u, \nabla v) = \sum_{k,j=1}^3 (D_ku_j, D_kv_j), \quad ||\nabla u|| = (\nabla u, \nabla u)^{\frac{1}{2}}
\]

(3)}
as scalar product and norm.

Moreover, we need the $B$-valued spaces $C^m(J, B)$ and $H^m(a, b, B)$, $m \in \mathbb{N}_0$, where $J \subset \mathbb{R}$ and $a, b \in \mathbb{R}$ ($a < b$), and where $B$ is any of the spaces above. In case of $C^0(\cdot)$ we simply write $C(\cdot)$, and we use $H, V, H^m, \ldots$ instead of $H(G), V(G), H^m(G), \ldots$, if the domain of definition is clear from the context.

Finally, let

$$P : L^2(G)^3 \longrightarrow H(G)^3$$

(7)

denote the orthogonal projection. Then it holds the decomposition

$$L^2(G)^3 = H(G)^3 \oplus \{ v \in L^2(G)^3 | v = \nabla p \text{ for some } p \in H^1(G) \},$$

into a direct sum of two closed subspaces [4], with means

$$(u, \nabla p) = 0 \text{ for all } u \in V(G)^3 \text{ and } p \in H^1(G).$$

(8)

### 2 An Implicit Euler Scheme

Because the projection $P$ from (7) commutes with the strong time derivative $\partial_t$, from the Stokes equations (1) we obtain the following evolution equations for the function $t \rightarrow v(t) \in H(G)^3$:

$$\partial_t v(t) - \nu \Delta v(t) = PF(t) \ (t \in (0, T)), \quad v(0) = v_0.$$  

(9)

In this case, the condition div $v = 0$ and the boundary condition $v = 0$ on $S$ are satisfied in the sense that we require $v(t) \in V(G)^3$ for all $t \in (0, T)$. Concerning the solvability of the evolution equations (9), the following result is well-known ([7], [10]):

**1. Proposition:** Let

$$v_0 \in H^2(G)^3 \cap V(G)^3, \quad F \in H^1(0, T, H(G)^3).$$

(10)

Then there is a unique solution $v$ of (9) in $(0, T) \times G$ such that

$$v \in C((0, T], H^2(G)^3 \cap V(G)^3), \quad \partial_t v \in C([0, T], H(G)^3) \cap L^2(0, T, H^1(G)^3).$$

(11)

Moreover, there is some constant $K$ depending only on $G, \nu, F, v_0$ and not on $t \in [0, T]$ such that for all $t \in [0, T]$

$$\int_0^t ||\nabla \partial_\nu v(\sigma)||^2 d\sigma \leq K, \quad ||v(t)||_2 \leq K, \quad ||\partial_\nu v(t)|| \leq K.$$  

(12)

Let us now consider the discrete equations (3), (4) under the weaker assumptions

$$v_0 \in H(G)^3, \quad F \in L^2(0, T, H(G)^3).$$

(13)

Using $P$ as above and noting that $F = PF$, we obtain in $G$ for $k = 1, 2, \ldots, N$ the equations

$$(v^k - v^{k-1}) - h \nu \Delta v^k = \int_{(k-1)h}^{kh} F(t) dt, \quad v^0 = v_0.$$  

(14)

Concerning the solvability of this system we have the following result:
2. **Proposition:** Under the above assumptions (13) there is a unique solution

\[ v^k \in H^2(G)^3 \cap V(G)^3 \quad (k = 1, 2, \ldots, N) \]  

(15)

of (14)

This statement can be proved as follows: If we define the Stokes operator \( A \) to be the extension of \(-P\Delta\) in \( H(G)^3\), then its domain of definition \( D(A) \) is \( H^2(G)^3 \cap V(G)^3 \). Because \( \lambda = (\nu h)^{-1} > 0 \) belongs to the resolvent set of \(-A\), the equations

\[ v^k = (\lambda + A)^{-1} F^\lambda_{k-1}, \quad F^\lambda_{k-1} \in H(G)^3 \]  

(see (4)) are uniquely solvable with \( v^k \in H^2(G)^3 \cap V(G)^3 \), as asserted above [8].

To prove the convergence of the discrete equations (14) to the evolution equations (9) and to estimate the discretization error, we use the approach "stability + consistency \(\Rightarrow\) convergence", see [5]. Let us define

\[ (\Pi v)(t_k) := v(t_k) - v(t_{k-1}) - \nu h P\Delta v(t_k), \quad (\Pi\{v^j\})(t_k) := v^k - v^{k-1} - \nu h P\Delta v^k. \]

Then the discretization error

\[ e^k := v^k - v(t_k) \]  

(16)

satisfies the identity

\[ e^k - e^{k-1} - \nu h P\Delta e^k = (\Pi\{v^j\})(t_k) - (\Pi v)(t_k) =: R^k, \]  

(17)

which is used to obtain estimates of \( e^k \) in terms of the right hand side \( R^k \) (\(\approx\) stability). Then the behavior of

\[ R^k = \int_{(k-1)h}^{kh} (\partial_t v(t) - \nu P\Delta v(t))dt - \{v(t_k) - v(t_{k-1})\} - \mu P\Delta v(t_k) \]

\[ = \int_{(k-1)h}^{kh} -\nu P\Delta (v(t) - v(t_k))dt \]

\[ =: -\nu P\Delta E^k \]  

(18)

as \( h \) tends to zero (\(\approx\) consistency) follows from the regularity properties of the exact solution of the Stokes equations (9), as stated in Proposition 1.

3 **Main Results**

To prove optimal convergence behavior of the approximate solution in Proposition 2 to the continuous solution of the Stokes equations in Proposition 1 it is crucial to use the integral

\[ h^{-1} \int_{(k-1)h}^{kh} F(t)dt \]

over the exterior force density on the right hand side in (2). Using the simpler term \( F(t_k) = F(kh) \) instead, the following optimal result cannot be proved.
**Theorem:** Let $T > 0$, $N \in \mathbb{N}$, and $G \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $S$ of class $C^2$. Assuming (10), let $v$ and $v^k (k = 1, 2, \ldots, N)$ denote the solution of (9) and (14), respectively. Then the discretization error $e^k$ (see (16)) satisfies the following estimates:

$$
||e^k||^2 + \sum_{j=1}^k (h \nu ||\nabla e^j||^2 + ||e^j - e^{j-1}||^2) \leq Kh^2,
$$

$$
||\nabla e^k||^2 + \sum_{j=1}^k (2(h \nu)^{-1}||e^j - e^{j-1}||^2 + \frac{1}{2}||\nabla (e^j - e^{j-1})||^2) \leq Kh.
$$

Here the constant $K$ depends only on $G, \nu$, and the data. Moreover, we even have uniform convergence with respect to the $H^2$-norm:

$$
\max \{||e^k||_2 \mid k = 1, 2, \ldots, N\} = o(1) \text{ as } h \to 0 \text{ or } N \to \infty.
$$

**Proof:** From (17) and (18) we obtain for the defect $e^k$ the identity

$$
(e^k - e^{k-1}) - h\nu P \Delta e^k = -v P \Delta e^k.
$$

Multiplying (19) scalar in $L^2$ by $2e^k$ and using (5) we find

$$
||e^k||^2 - ||e^{k-1}||^2 + ||e^k - e^{k-1}||^2 + 2h\nu ||\nabla e^k||^2 = 2\nu (\nabla E^k, \nabla e^k) \leq 2 (h \nu)^{\frac{1}{2}} ||\nabla e^k|| (h^{-1} \nu)^{\frac{1}{2}} ||\nabla E^k|| \leq h \nu ||\nabla e^k||^2 + h^{-1} \nu ||\nabla E^k||^2 := S_1 + S_2.
$$

Because of

$$
S_2 = h^{-1} \nu \left\| \int_{(k-1)h}^{kh} \int_{(k-1)h}^{kh} \partial_\sigma \nabla v(\sigma) d\sigma dt \right\|^2 \leq \nu \int_{(k-1)h}^{kh} \left\| \partial_\sigma \nabla v(\sigma) \right\|^2 dt \leq \nu h \int_{(k-1)h}^{kh} ||\nabla v(\sigma)||^2 d\sigma,
$$

it follows

$$
||e^k||^2 - ||e^{k-1}||^2 + ||e^k - e^{k-1}||^2 + h \nu ||\nabla e^k||^2 \leq \nu h \int_{(k-1)h}^{kh} ||\partial_\sigma \nabla v(\sigma)||^2 d\sigma \quad (20)
$$

for all $k = 1, 2, \ldots, N$. Thus summing up over $k$, using $||e^0||^2 = 0$ and (12), the first estimate of the theorem is proved.

To prove the second estimate we multiply (19) scalar in $L^2$ by $2(e^k - e^{k-1})$. Here we obtain

$$
2||e^k - e^{k-1}||^2 + 2h\nu (\nabla e^k, \nabla (e^k - e^{k-1})) \leq 2||e^k - e^{k-1}||^2 + h\nu ||\nabla e^k||^2 + ||\nabla (e^k - e^{k-1})||^2.
$$

It follows

$$
2 \cdot \frac{h \nu}{2} \left( ||\nabla (e^k - e^{k-1})|| \cdot (2h^{-1} \nu)^{\frac{1}{2}} ||\nabla E^k|| \right.
$$

$$
\leq \frac{h \nu}{2} ||\nabla (e^k - e^{k-1})||^2 + 2h^{-1} \nu ||\nabla E^k||^2 =: S_3 + 2S_2.
$$
Using the above estimate for $S_2$ again, it follows

$$2\|e^k - e^{k-1}\|^2 + h\nu(\|\nabla e^k\|^2 - \|\nabla e^{k-1}\|^2) - \frac{h\nu}{2} \|\nabla(e^k - e^{k-1})\|^2$$

\[
\leq 2\nu h^2 \int_{(k-1)h}^{kh} \|\partial_\sigma \nabla v(\sigma)\|^2 d\sigma,
\]

hence

$$\|\nabla e^k\|^2 - \|\nabla e^{k-1}\|^2 + 2(h\nu)^{-1}\|e^k - e^{k-1}\|^2 + \frac{1}{2} \|\nabla(e^k - e^{k-1})\|^2$$

\[
\leq 2h \int_{(k-1)h}^{kh} \|\partial_\sigma \nabla v(\sigma)\|^2 d\sigma,
\]

which implies the second estimate of the theorem.

Finally, we want to prove convergence with respect to the $H^2$-norm. From (19) we conclude

$$P\Delta e^k = (h\nu)^{-1}(e^k - e^{k-1}) + h^{-1}P\Delta E^k,$$

which implies

$$\|P\Delta e^k\|^2 \leq 2(h\nu)^{-2}\|e^k - e^{k-1}\|^2 + 2h^{-2}\|P\Delta E^k\|^2. \quad (21)$$

Using (11), we find the following estimate for the second term:

$$2h^{-2}\|P\Delta E^k\|^2 \leq 2\nu h \int_{(k-1)h}^{kh} \|P\Delta v(t) - v(t_k)\|^2 dt$$

\[
\leq 2 \max_{\sigma, \tau \in [0, T]} \|P\Delta v(\sigma) - v(\sigma)\|_2
\]

\[
= o(1) \quad \text{as} \quad h \to 0.
\]

Thus it remains to show that also the first term of (21) tends to zero. Defining

$$T^k := \frac{(e^k - e^{k-1})}{h} \quad (k = 1, 2, \ldots, N)$$

for abbreviation, from (19) we obtain the identity

$$T^k - T^{k-1} = h\nu P\Delta T^k$$

\[
= -h^{-1}\nu P\Delta \left\{ \int_{(k-1)h}^{kh} (v(t) - v(t_k)) dt - \int_{(k-2)h}^{(k-1)h} (v(t) - v(t_{k-1})) dt \right\}
\]

\[
= -h^{-1}\nu P\Delta G^k,
\]
where $G^k$ is defined by the above term in brackets.

Now scalar multiplication in $L^2$ by $2T^k$ yields as above

$$||T^k||^2 - ||T^{k-1}||^2 + ||T^k - T^{k-1}||^2 + 2h\nu||\nabla T^k||^2 \leq h\nu||\nabla T^k||^2 + h^{-3}\nu||\nabla G^k||^2,$$  \hspace{1cm} (22)

hence

$$||T^k||^2 - ||T^{k-1}||^2 + ||T^k - T^{k-1}||^2 + h\nu||\nabla T^k||^2 \leq h^{-3}\nu||\nabla G^k||^2.$$  

Because

$$G^k = -\int_{(k-1)h}^{kh} \int_t^t (\partial_{\sigma} v(\sigma) - \partial_{\sigma} v(\sigma - h)) d\sigma dt,$$

we find the estimate

$$||\nabla G^k||^2 \leq h^3 \int_{(k-1)h}^{kh} ||\partial_{\sigma} \nabla (v(\sigma) - v(\sigma - h))||^2 d\sigma .$$

Thus from (22) we obtain

$$||T^k||^2 - ||T^{k-1}||^2 + ||T^k - T^{k-1}||^2 + h\nu||\nabla T^k||^2 \leq \nu \int_{(k-1)h}^{kh} ||\partial_t \nabla (v(t) - v(t - h))||^2 dt ,$$

and summing up yields

$$||T^k||^2 + \sum_{j=2}^k (||T^j||^2 - ||T^{j-1}||^2 + \nu h||\nabla T^j||^2) \leq ||T^1||^2 + \nu \int_{0}^{T} ||\partial_t \nabla (v(t) - v(t - h))||^2 dt \hspace{1cm} (23)$$

$$= o(1) \text{ as } h \to 0 ,$$

because the integral vanishes as $h \to 0$, and because by (20) (note $||e^0|| = 0$)

$$||T^1||^2 = ||(e^1 - e^0)h||^2 \leq \nu \int_{0}^{h} ||\partial_t \nabla v(t)||^2 dt = o(1) .$$

Thus (23) implies that also the first term of (21) tends to zero as $h \to 0$, hence $||P\Delta e^k||^2 = o(1)$ as $h \to 0$, and the asserted convergence with respect to the $H^2$-norm follows by means of Cattabriga's estimate [1]. This proves the theorem.
References


