Abstract

Internal and inertial waves are ubiquitous in the world’s oceans, and the energy that they transfer from the global tide to the smaller scales may impact the vertical mixing and global currents. The dynamics of internal and inertial waves in closed domains possesses a remarkable property of self-focusing on geometrical limit cycles called wave attractors. The significant growth of wave amplitude at the wave attractors results in instabilities for viscous fluids. The scenarios of the transition to turbulence and the description of the fully turbulent regimes differ substantially from the cases observed in closed domains in the absence of wave attractors. Our previous studies demonstrated the key role of a cascade of triadic resonances as the route to fully developed wave turbulence, either with overturning events or not. Presently we consider a shallow trapezoid and the wave attractors that form after multiple reflections from the horizontal boundaries. The saturation time of (n,1) wave attractors in laminar viscous fluids grows linearly with n if the wavemaker is located at the sidewall. Next we show that in a shallow elongated domain with small aspect (depth-to-length) ratio, the frequency spectrum of (1,1) wave attractor motion may exhibit significant peaks at integer and half-integer multiples of the forcing frequency. For an aspect ratio of about one tenth the temporal average of the total kinetic energy grows monotonically with the amplitude and exhibits a bend at a particular amplitude. Below this amplitude the cascade transferring energy to integer superharmonic components prevails, while above it the amplitudes of waves at integer and half-integer multiples of the forcing frequency are comparable. The spatial spectra of the waves are compared for domains of aspect ratio varying from small values to values close to unity. It is shown that in the former case (i.e. for elongated shallow domains) the spectrum exhibits two zones at small and high wave numbers characterized by different slopes. The fully turbulent regimes show a trend toward a long-term evolution leading to new regimes with a complex resonant dynamics of large-scale coherent structures.

Keywords: internal waves, inertial waves, wave attractors, wave turbulence

1 Introduction

Internal and inertial waves are responsible for energy transport in the worlds oceans, as well as in planets and other astrophysical objects. May be even more important are the effects of large waves overturning and subsequent mixing in oceans, since they may affect the vertical stratification profile, and in such a way internal waves may influence the global currents, which are shaped according to stratification profiles. Internal waves possess a property, which makes them remarkably different from conventional acoustic or electromagnetic waves: the wave beams after reflection from a surface conserve the angle with the gravity, rather than the angle with the vertical to the surface. In the late 1990s it was found that due to just geometrical properties of billiard with such a rule we can observe phenomenon of wave attractors - limit cycles for the wave beams [1]. These discoveries provoked a number of studies of instabilities of flows associated with wave attractors in viscous fluids. It was found, that cascades of triadic resonances instabilities (TRI) are responsible for transition to the wave turbulence, assuming the vertical and horizontal scales of motion are of the same order. On the other hand, the ratio of horizontal to vertical scales of wave motions in the oceans may vary from 1 to 10 (which is quite typical for tidally generated internal waves) and higher. This is why it is important to study the specifics of instabilities for large-aspect ratio wave motions, or in other words, for small (but not too small, so hydrostatic approximation still can not be applied) hydrostatic parameter.
1.1 Laboratory/mathematical setup

To study the large-aspect ratio effects we will first of all follow the "classical" setup to study of wave attractors in the laboratory as was given in [2, 3, 4, 5, 6, 7], the sketch is given in Fig. 1. The stratified fluid is confined in trapezoid container with two horizontal boundaries, the left boundary is vertical and the right wall is inclined with respect to the vertical by the angle $\alpha$. Previous studies of (1,1) waves attractors mostly considered the height and length to be roughly of the same order, e.g. $H = 40$ cm and $L = 1.5H = 60$ cm. Below we keep $H = 40$ cm, and $L$ will be equal either to 60 cm or 300 cm.

Since the dimensional analysis is a subject of debate for such a laboratory setup, we will stick to dimensional setup (in CGS units) while giving all the important nondimensional parameters.

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla) \vec{v} = -\frac{1}{\rho_m} \nabla p + \rho_s \vec{g} + \nu \Delta \vec{v} \]  

(1)
\( \nabla \cdot \vec{v} = 0 \). \hspace{1cm} (2)

where \( \rho = \rho_m + \rho_s \), \( \rho_m \) is the minimal density (density of “fresh” water), and \( \rho_s \) the partial density of the dissolved salts, subject to the following equation:

\[
\frac{\partial \rho_s}{\partial t} + \vec{v} \cdot \nabla \rho_s = \nabla \nu \frac{\nu}{\text{Sc}} (\nabla \rho_s), \hspace{1cm} (3)
\]

Sc is the Schmidt number, so \( \nu/\text{Sc} \) corresponds to the salts diffusivity.

The energy is injected to the system via the periodically oscillating upper boundary. Since the amplitude of the oscillation is small compared to the height (\(< 0.01\ H\) we will model the oscillations with the help of the boundary condition on the vertical velocity:

\[
\vec{v} |_{y=H} = a \omega_0 \sin(\omega_0 t) \sin(\pi x/L) \vec{e}_y. \hspace{1cm} (4)
\]

On all the other boundaries we have no-slip condition. The equations (1, 2, 3, 4) together with no-slip boundary conditions form a well-posed initial boundary value problem.

Direct numerical simulation of turbulent regimes in the described domain can be challenging because of non-orthogonal geometry. Spectral elements approach \([9, 10]\) allows to overcome these difficulties. In this paper we will stick to the same approach for numerical simulation as given in \([3]\).

\[\text{Figure 5: Averaged full kinetic energy as function of amplitude for } \varepsilon = 0.13 \text{ (left) and relative kinetic energy (right) normalised by maximum kinetic energy at the wavemaker. Amplitude (horizontal axis) is given in centimeters. The relative kinetic energy experiences a jump at an amplitude, for which there is a growth of wave at integer multiples of the forcing half-frequency.}\]

1.2 Non-dimensional parameters

In case of an ideal fluid the mathematical setup described above involves five dimensional parameters (\(\omega, N, \alpha, H, L\)). With the help of a horizontal translation followed by horizontal/vertical compression they can be transformed to a set of just two parameters: \((d, \tau)\) \([1]\). \(d\) and \(\tau\) are nondimensional, the first corresponds to the location of the slope, and second is the nondimensional
height after vertical compression in such a way that the wavebeams have an angle 45° with the gravity, so it involved external frequency.

For a viscous and diffusive flow there appear additional dimensionless parameters. The first is $\bar{Re} = Ha\omega_0/\nu$. Though it may be interpreted as an analogy to the number of Reynolds, it does not play exactly the same role as in the shear flows like pipe or Couette flows. Due to the ability of the attractor to accumulate the kinetic energy, the shear velocity can grow by orders of magnitude as compared to the velocity at the wavemakers, hence the difficulty of the use of $\bar{Re} = Ha\omega_0/\nu$ as a measure of the shear motion in this study. The second parameter is the aspect ratio $L/H$, it is inverse to $\varepsilon = H/L$, which is often referred to as the “hydrostatic” nondimensional parameter, if H and L are also the scales of the horizontal and vertical velocities [11].

2 Dynamics of the total kinetic energy

From the point of view of an ideal fluid there is no difference whether the domain has a shape where the horizontal and vertical sizes are similar, or if they differ by orders. The ray-theory solution depends on just two parameters ($d, \tau$) mentioned in the previous section.

For real (viscous) fluids the aspect ratio $\varepsilon = H/L$ may play a significant role. In Figures 2,3 snapshots of the field of vertical velocity are shown for $\varepsilon = 0.66$ and $\varepsilon = 0.13$ after wave attractor regime was fully established. For each aspect ratio the two pictures are shifted in time by the quarter of period to show the typical cases of waves over the attractors.

In what follows we shall discuss two forms of kinetic energy: full and relative. The former is described by formula:

$$E = \frac{1}{V} \iiint_V \frac{1}{2} v^2 dV$$

(5)

The effects of density variations on computation of total kinetic energy are small for our comparative study, so we will not take it into account. The relative kinetic energy is defined by the ratio:

$$E = \frac{2}{(a\omega_0)^2} E,$$

(6)

the full kinetic energy is normalized here by the maximum kinetic energy at the wavemaker [12].

Dynamics of full kinetic energy in case of $\varepsilon = 0.66$ (left) and $\varepsilon = 0.13$ (right) is shown in Figure 4.

These plots allow to conclude that amplitude of pulsations and the saturation (measured in forcing periods) time both decrease with increase of aspect ratio.
The full kinetic energy increases monotonically with growth of the forcing amplitude as shown in Fig. 5. The red squares correspond to dimensional full kinetic energy, averaged over 10 last periods $T_0$. It can be seen that the temporal average of total kinetic energy grows monotonically with amplitude, but have a bend at a particular amplitude $a_{cr} \sim 0.14$ cm. In the next section we show that below this amplitude the cascade transferring energy to superharmonic components prevails, while above this forcing amplitude the wave amplitudes at integer multiples of $f_0$ integer multiples of $f_0/2$ are comparable. RHS of Fig. 5 shows that if the averaged total energy is normalized by the maximal energy at the wave maker (as in [12]) then the plot of such an averaged relative kinetic energy experiences a jump at the amplitude $a_{cr}$.

Longtime dynamics of total kinetic energy for large aspect ratio ($\varepsilon = 0.13$) is shown in Fig. 6.
Here we see the difference between longtime behaviour for large aspect ratio in turbulent regimes of moderate external forcing: the first cascade of resonances results in “primary” turbulence in internal between about $10T_0$ and $60T_0$. After that there is an accumulation of small scale turbulence to larger coherent structures with specific frequency spectra.

### 3 Time-frequency analysis

Temporal spectral analysis of large aspect ratio regimes reveals the new phenomena as compared to previous studies: even for the low amplitudes we reveal the superharmonic cascades (Fig.8). Up to $a = 0.12$ cm this superharmonical scenario prevails over other instabilities, and beginning with $a = 0.14$ cm one can see the rise of motion at odd multiples of the half-frequency of external forcing. For larger external forcing the amplitudes of the waves at these ‘half-frequencies’ begin to equate with those at integer superharmonic frequencies. And of course one can note that both integer and half-integer type of waves are saturated until the buoyancy frequency (marked by black line in figures). Beyond the buoyancy frequency an exponential fading takes place as can be predicted by the linear dispersion relation. The relevance of the dispersion relation will also be shown below with the help of $(\omega - \theta)$ diagrams [13], where \(\theta\) is the angle between a wave and the gravity direction.

In Fig. 11 the process of growth of subharmonics to the level of superharmonics is illustrated with the help of time-frequency diagrams. At small amplitudes the only line is visible at the forcing frequency. For higher amplitudes there are more frequencies, and one can see the time development of the instabilities.

![Time-frequency diagrams for increasing amplitudes](image)

Figure 11: Time-frequency diagrams for increasing amplitudes (left to right) $a = 0.00001, 0.01, 0.14, 0.22$ cm in case of large aspect ratio domain ($\varepsilon$ = 0.33)

Another way to show if the instabilities provokes a regime which still obeys the linear dispersion relation is the $\omega - \theta$ diagram [13]. The Fig. 12 shows the point off maximum amplitudes in $\omega - \theta$ plane, $\theta$ being the angle of propagation of plane waves. This picture supports the weak turbulence scenario described above since all the secondary waves suit perfectly the linear dispersion relation.
Conclusions

We have shown that taking into account of the aspect ratio is very important for estimation of the behavioural patterns in nonlinear development of viscous internal wave regimes. Our results allow to widen the TRI scenario proposed in [4]: if in frequency domain the interval between the forcing frequency and buoyancy frequency contains superharmonics of the forcing frequency, then the following scenario of transition to turbulence is taking place, as the forcing amplitude increases:

1. first, there is an energy pumping to the integer superharmonics between the forcing frequency and buoyancy frequency;
2. next, there happens a hydrodynamical instability at multiples of half-frequency,
3. and finally we have triadic resonances happening in every multiple of half-frequency.

References


