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DYNAMIC MODE DECOMPOSITIONS OF PHONATION ONSET – COMPARISON OF DIFFERENT METHODS

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Abstract

Four dynamic mode decomposition (DMD) methods are used to analyze a simulation of the phonation onset carried out by in-house solver based on the finite element method. The dataset consists of several last periods of the flow-induced vibrations of vocal folds (VFs).

The DMD is a data-driven and model-free method typically used for finding a low-rank representation of a high-dimensional system. In general, the DMD decomposes a given dataset to modes with mono-frequency content and associated complex eigenvalues providing the growth/decay rate that allows a favourable physical interpretation and in some cases also a short-term prediction of system behaviour. The disadvantages of the standard DMD are non-orthogonal modes and sensitivity to an increased noise level which are addressed by following DMD variants. The recursive DMD (rDMD) is an iterative DMD decomposition producing orthogonal modes. The total least-square DMD and the higher order DMD (hoDMD) are methods substantially reducing a high DMD sensitivity to noise.

All methods identified very similar DMD modes as well as frequency spectra. Substantial difference was found in the real part of the spectra. The final dataset reconstruction is the most accurate in the case of the recursive variant. The higher order DMD method also outperforms the standard DMD. Thus the rDMD and the hoDMD decompositions are promising to be used further for the parametrization of a VF motion.

Keywords: dynamic mode decomposition, recursive DMD, total least-square DMD, higher order DMD, flow-induced vibrations.

1 Introduction

The phonation onset or generally human phonation process is a complex, highly interesting phenomenon which is not fully understood yet, [1, 2, 3]. Due to the very limited accessibility of vocal tract, numerical modelling of flow-induced vibrations of vocal folds (VFs) plays an irreplaceable role. Nevertheless, accurate numerical simulation of coupled fluid-structure interaction problems is computationally expensive. Therefore, any way to decrease the simulation computational cost is highly desired. In this contribution, we study the dynamic mode decomposition (DMD) and its variants as promising tools for parametrization of VFs motion.

The DMD is a technique for determining, directly from data, both a low-dimensional subspace and a description of the dynamics on that subspace, see [4]. DMD decomposes a given dataset to modes and associated eigenvalues. The DMD modes have mono-frequency content similar to modal analysis. Additionally, the DMD eigenvalues are complex, providing also the growth/decay rate of the particular spatial mode. Thus, DMD provides a better physical insight than the commonly used proper orthogonal decomposition (POD) or similar model order reduction (MOR) methods, see [5]. The DMD is well theoretically established due to connection with the Koopman operator and it enables to shortly forecast system behaviour, a feature heavily used in flow control, [6]. The disadvantage of the DMD is its high sensitivity to the noise level present in the analyzed dataset.

An interesting DMD variant – recursive DMD (rDMD) [7], tries to merge advantages of POD (modes orthogonality, low truncation error) together with the DMD advantages (mono-frequency modes). The orthogonality of resulting DMD modes is ensured by an iterative (greedy) DMD usage on the dataset residual. Due to the spatial modes orthogonality the rDMD is suitable for application in projection-based MOR methods. Furthermore, study [7] verified low reconstruction error of rDMD on the transient cylinder wake. Another DMD variant – total least-square DMD

(tlsDMD) [8], shows a way how to systematically decrease the DMD sensitivity to noise. It is based on the total least-square approach applied within the DMD framework, and it is useful for experimental data analysis, see [9]. Finally, the higher order DMD (hoDMD) is another generalization of the DMD method, see [10]. It is partially similar to the extended DMD method, see [11], in the fact that it is applied on an enlarged input matrix, here given by d -times repeated learning dataset. Nevertheless, due to a different, generalized, fundamental DMD relation between the subsequently ordered snapshots, the hoDMD suppresses the DMD noise sensitivity very well. The robustness of hoDMD was examined by e.g. flight flutter testing [10].

All four DMD methods are here compared on the dataset obtained by a FSI in-house solver based on the finite element method, see [12]. The selected structural part of the FSI simulation is composed of several periods of flow-induced VFs vibrations. Since the analyzed phenomenon is unstable, the DMD reconstruction have to carefully determine growth rates, see [13]. This is an important difference compared to work [14], where DMD methods were utilized for measurement of the flow over a membrane-wing at constant angle of attack as this process by nature cannot possess unstable eigenvalues. Comparison of multiple DMD approaches can provide us more confidence in the growth rates computation.

Furthermore, the DMD was used in the paper [5] on the coupled problem of the flow over flag where they successfully identified the driving mechanism of the chaotic flapping onset. The key ingredient was to incorporate both involved physical fields into the analysis what required introduction of a meaningful FSI norm related to the energy of the FSI system. Nevertheless, the introduction of meaningful FSI norm in our case is possible (flow data is available) but complicated due to need to balance energy of the structural and the fluid flow part. If the FSI norm is not chosen properly one physical field would dominate in the energy norm and the DMD analysis would not provide any additional insight. The DMD analysis of FSI results in [14] indicate that considering even relative well balanced results of both physical fields does not have to substantially improve results. Therefore we leave this option as an interesting direction for future research.

In the FSI context an additional problem can arise due to a time-dependent FSI domain and its representation by the immersed-boundary method, see [15]. The affiliation of points in a FSI computational mesh is changing from the structural to the fluid domain and vice versa and it makes the direct application of DMD entangled. Therefore authors of [15, 5] projected the flow results around moving body to the rotating window firmly connected with that body before DMD application. Nevertheless, this issue is not related to our solver using ALE method in the fluid flow part and the reference coordinates in the structure part description, see [12].

The paper is organised as follows: The DMD method is introduced in the first section. The second section contains description of three DMD variants. Then all DMD methods are compared on the flow-induced VFs vibrations in the third section followed by the final conclusion.

2 Dynamic mode decomposition

First, let us assume a general ODE system in the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_1) = \mathbf{x}_{\text{init}}, \quad (1)$$

with a continuous, possibly nonlinear function $\mathbf{f}(\mathbf{x}, t)$ and with solution $\mathbf{x}(t) \in \mathbb{R}^J$ for $t \in \langle t_1, t_2 \rangle$. Such a system can be obtained also from a spatial semi-discretization of partial differential equations (PDE), see [11]. Next, let us denote solution of (1) at time instants $t_k = k \cdot \Delta t$, $k = 1, 2, \dots$ as

$$\mathbf{x}_k = \mathbf{x}(t_k). \quad (2)$$

The central point of the DMD method is to assume that the time development of system (1) at discrete times t_k is given as a linear evolution from the previous states, i.e. $\mathbf{x}_{k+1} \approx \mathbb{A}\mathbf{x}_k$. Rearranging this relation into the matrix form for the given dataset of approximate solutions $\mathcal{D} = \{\mathbf{x}_i\}_{i=1}^N$ and unknown matrix $\mathbb{A} \in \mathbb{R}^{J \times J}$ with J denoting the number of components of vector $\mathbf{x}$ leads to

$$\mathbb{X}_2^N \doteq \mathbb{A}\mathbb{X}_1^{N-1}, \quad (3)$$

where the columns of matrix $\mathbb{X}_l^m$, $l < m \in \mathbb{N}$ are composed of vectors $\mathbf{x}_l, \mathbf{x}_{l+1}, \dots, \mathbf{x}_{m-1}, \mathbf{x}_m$. If DMD is applied on a solution of PDE the vector dimension J is usually substantially larger than N , i.e. the number of snapshots $\mathbf{x}(t_k)$ in the dataset $\mathcal{D}$.

Algorithm. Here, we present DMD algorithm as given in [11]. In this case the full DMD matrix $\mathbb{A} \in \mathbb{R}^{J \times J}$ is approximated by $\mathbb{A} = \mathbb{U} \tilde{\mathbb{A}} \mathbb{U}^T$, where the matrix $\tilde{\mathbb{A}} \in \mathbb{R}^{r \times r}$ is the projection of $\mathbb{A}$ onto the space formed by the first r left-singular vectors of matrix $\mathbb{X}_1^{N-1}$ obtained by singular value decomposition (SVD), i.e. $\tilde{\mathbb{A}} = \mathbb{U}^T \mathbb{A} \mathbb{U}$. The steps of the DMD are following:

1. Perform **truncated** SVD of $\mathbb{X}_1^{N-1} \approx \mathbb{U} \Sigma \mathbb{V}^T$, $\mathbb{U} \in \mathbb{R}^{J \times r}$, $\mathbb{V} \in \mathbb{R}^{N-1 \times r}$ and $\Sigma \in \mathbb{R}^{r \times r}$ with $\sigma_i = \Sigma_{ii}$ are singular values of $\mathbb{X}_1^{N-1}$. Here r is chosen such that root mean square (RMS) error $\sqrt{\frac{\sigma_{r+1}^2 + \sigma_{r+2}^2 + \dots + \sigma_{N-1}^2}{\sigma_1^2 + \sigma_2^2 + \dots + \sigma_{N-1}^2}}$ is smaller than tunable threshold ϵ_{SVD} . For precise data the recommended value is $\epsilon_{\text{SVD}} = 10^{-8}$, but larger values may be convenient to filter out numerical and experimental errors, see [10].
2. Construct matrix $\tilde{\mathbb{A}}$ as $\tilde{\mathbb{A}} = \mathbb{U}^T \mathbb{X}_2^N \mathbb{V} \Sigma^{-1} \in \mathbb{R}^{r \times r}$, where the diagonal matrix Σ contains r largest singular values of $\mathbb{X}_1^{N-1}$.
3. Compute eigendecomposition of $\tilde{\mathbb{A}}$, i.e. find a diagonal matrix Λ and matrix $\mathbb{W} \in \mathbb{C}^{r \times r}$ such that $\tilde{\mathbb{A}} \mathbb{W} = \Lambda \mathbb{W}$. Here we assume that the matrix $\tilde{\mathbb{A}}$ is diagonalizable thus each eigenvalue of matrix $\tilde{\mathbb{A}}$ has multiplicity one.
4. Compute the DMD modes Φ_i as column of the matrix $\Phi = \mathbb{X}_2^N \mathbb{V} \Sigma^{-1} \mathbb{W}$. The eigenvalues of $\tilde{\mathbb{A}}$ are the same as $\mathbb{A}$, see [11]. Determine mode participation coefficients b_i for each DMD mode Φ_i from equation $\mathbf{b} = \Phi^\dagger \mathbf{x}_1$, where $\Phi^\dagger$ denotes the Moore-Penrose pseudo-inverse of the matrix Φ .
5. Finally use DMD modes, eigenvalues and mode participation coefficients for the reconstruction of the given dataset as described in the next paragraph by equation (4).

Note. The mode participation coefficients in step 4 can be computed alternatively by the least-square approach as suggested in [10]. However this approach is computationally demanding.

Reconstruction. Once the DMD matrix $\mathbb{A}$ or more precisely the DMD matrix approximation $\tilde{\mathbb{A}}$ is known/constructed, its eigendecomposition on (nonzero) eigenvalues $\lambda_i \in \mathbb{C}$ and corresponding eigenvectors $\Phi_i \in \mathbb{C}^J$ provides complete dynamical information about the approximating linearized system. The analyzed state-space trajectory can be reproduced according to the formula

$$\mathbf{x}_{k+1} \approx \mathbf{x}_{k+1}^{\text{rec}} = \tilde{\mathbb{A}}^k \mathbf{x}_1 = \sum_{i=1}^r b_i \Phi_i \lambda_i^k = \sum_{i=1}^r b_i \Phi_i \exp(\omega_i t_k), \quad (4)$$

where $b_i \in \mathbb{C}$ are the coefficients of i -th DMD mode participation and $\omega_i \in \mathbb{C}$ are approximate continuous-time eigenvalue spectrum given by $\omega_i = \ln(\lambda_i)/\Delta t$. The real and the imaginary part of ω_i give information about the exponential growth/decay rate and the frequency content of the particular mode, respectively. For real input dataset $\mathcal{D}$ only the real part of the reconstruction $\mathbf{x}_{k+1}^{\text{rec}}$ is used, i.e. $\mathbf{x}_{k+1} \approx \text{Re}(\mathbf{x}_{k+1}^{\text{rec}})$.

Note. In practice the s most dominant (from available r) DMD modes are selected. Although there are more criteria available we choose such modes that maximize the magnitude of modal amplitude during the whole analyzed time interval, which is computed as $z_i = \sqrt{\sum_{k=1}^{N-1} (b_i \exp(\omega_i t_k))^2}$.

3 Variants of DMD

Three additional methods based on the previously presented DMD are described.

3.1 Recursive DMD

The recursive DMD (rDMD) is based on the idea to improve the DMD by an important POD advantage, namely by the orthogonality of resulting modes, [7]. This is achieved by an iterative (greedy) algorithm, where in each step only one real DMD mode is chosen such that minimizes average RMS reconstruction error, see [7]. After selection of this new DMD mode, the dataset is modified in a way that the entire trajectory of the just selected one DMD mode is subtracted from old dataset. This ensures the orthogonality of the DMD modes.

On the other hand the formula (4) is no more valid. The DMD modes still have a mono-frequency content; however, their time behaviour does not follow the formula $\exp(\omega_i t_k)$ any more. For this reason the authors of [7] consider the rDMD application to be more suitable for general transient simulations than for purely periodic phenomena.

3.2 Total least-square DMD

The total least-square DMD (tlsDMD) addresses the problem of the DMD sensitivity to noise, [8]. The fundamental DMD relation $\mathbf{x}_{k+1} \approx \mathbb{A}\mathbf{x}_k$ is equivalent to minimizing the residual e_{k+1} in $\mathbf{x}_{k+1} + e_{k+1} = \mathbb{A}\mathbf{x}_k$. However, this implies that no residual is present in the snapshot $\mathbf{x}_k$. Study [9] suggested to consider both residuals as $\mathbf{x}_{k+1} + e_{k+1} = \mathbb{A}(\mathbf{x}_k + e_k)$ and find the matrix $\mathbb{A}$ that minimizes both of them simultaneously, see also [8]. This leads to an algorithm as given in [9]:

1. Assembly input matrix $\mathbb{Z} = \begin{pmatrix} \mathbb{X}_1^{N-1} \\ \mathbb{X}_2^N \end{pmatrix}$. Apply truncated SVD on $\mathbb{Z}$ that yields $\mathbb{Z} \approx \mathbb{U}\Sigma\mathbb{V}^T$, $\mathbb{V} \in \mathbb{R}^{N-1 \times r}$.
2. Project matrices $\mathbb{X}_1^{N-1}, \mathbb{X}_2^N$ on the subspace of r right-singular vectors of $\mathbb{Z}$, i.e. form matrices $\tilde{\mathbb{X}}_1 = \mathbb{X}_1^{N-1}\mathbb{V}$ and $\tilde{\mathbb{X}}_2 = \mathbb{X}_2^N\mathbb{V}$.
3. Calculate the (reduced) SVD of $\tilde{\mathbb{X}}_1 = \mathbb{U}_1\Sigma_1\mathbb{V}_1^T$.
4. Compute the approximate DMD matrix $\tilde{\mathbb{A}}$ as $\tilde{\mathbb{A}} = \mathbb{U}_1^T\tilde{\mathbb{X}}_2\mathbb{V}_1\Sigma_1^{-1}$. The DMD matrix is related to the approximate one by $\mathbb{A} = \mathbb{U}_1^T\tilde{\mathbb{A}}\mathbb{U}_1$.
5. Then proceed normally with step 3. of the standard DMD algorithm.

3.3 Higher order DMD

Another generalization of the DMD is called the higher order DMD (hoDMD), [10]. It replaces relation (3) by a more general one

$$\mathbf{x}_{k+d} \approx \mathbb{A}_1\mathbf{x}_k + \mathbb{A}_2\mathbf{x}_{k+1} + \dots + \mathbb{A}_d\mathbf{x}_{k+d-1} \quad (5)$$

where d is a suitably chosen parameter. The relation (5) can be interpreted such that a snapshot (i.e. one solution vector) is estimated by a linear combination of not only the previous one but rather of d previous snapshots. In practice dataset $\mathbb{X}_1^N$ is projected on r left-singular vectors denoted as $\tilde{\mathbb{X}}_1^N \in \mathbb{R}^{J \times r}$ before rewriting formula (5) into the matrix form $\hat{\mathbb{X}}_2^{k-d+1} = \hat{\mathbb{A}}\hat{\mathbb{X}}_1^{k-d}$, where

$$\hat{\mathbb{A}} = \begin{pmatrix} 0 & \mathbb{I} & 0 & \dots & 0 & 0 \\ 0 & 0 & \mathbb{I} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbb{I} & 0 \\ \tilde{\mathbb{A}}_1 & \tilde{\mathbb{A}}_2 & \tilde{\mathbb{A}}_3 & \dots & \tilde{\mathbb{A}}_{d-1} & \tilde{\mathbb{A}}_d \end{pmatrix} \quad \text{and} \quad \hat{\mathbb{X}}_1^{k-d+1} = \begin{pmatrix} \tilde{\mathbb{X}}_1^{k-d+1} \\ \tilde{\mathbb{X}}_2^{k-d+2} \\ \vdots \\ \tilde{\mathbb{X}}_d^k \end{pmatrix} \quad (6)$$

and $\mathbb{I}$ denotes the identity matrix. Again, the (projected) DMD matrix $\hat{\mathbb{A}}$ does not need to be explicitly assembled. Rather, it is constructed approximately in a manner similar to the standard DMD, for details and individual steps of the hoDMD algorithm see [10].

4 Numerical results

First, the numerical simulation of phonation onset performed by in-house solver based on the FEM is presented. Then different DMD methods are applied on this dataset and their results are compared.

4.1 Flow-induced vibrations

The flow-induced vibrations of VFs were simulated using the geometrical as well as the numerical configuration of FSI problem overtaken from [12]. Particularly, the initial gap between two opposite vocal folds was 0.8 mm and the time step was chosen as $\Delta t = 4 \cdot 10^{-5}$ s. Furthermore, a constant inlet airflow velocity $\mathbf{v}_{\text{Dir}} = (1.98, 0)$ m/s exceeding the critical flutter velocity $v_{\text{crit}} \approx 1.9$ m/s was prescribed.

Figure 1 shows time behaviour of the displacement monitored at one point from the top of bottom VF labeled as point *A* (for its position see Figure 2 right). The clear occurrence of flutter can be observed up to time $t = 0.2956$ s. The gradual decreasing of the gap is documented in Figure 1 right.

Figure 1: **Left:** VF displacement in x -direction monitored at point *A*. The time interval used for further analysis is highlighted. **Right:** Time behaviour of gap between the both vocal folds. Black line denotes the initial gap 0.8 mm.

4.2 DMD results

Here, we apply four DMD variants on the dataset consisting of 2500 time steps in time interval $\langle 0.176, 0.276 \rangle$ [s] of the numerical approximation of VF structural displacements, see Figure 1. This time interval is substantially larger than the recommended minimal length of one and half period of dominant frequency, see [10]. The requirement of Δt being at least five times smaller than the smallest period of interest leads to theoretical resolution limit of $f < 5$ kHz.

The threshold $\epsilon_{\text{SVD}} = 3 \cdot 10^{-8}$ is chosen in truncated SVD, which is the first step of all the DMD variants. This value follows the recommendation of [10], i.e. ϵ_{SVD} to be comparable to the noise level observable from logarithmic graph of singular values. Finally, it leads to the dimension of SVD subspace of $r \sim 500$. The hoDMD parameter $d = 100$ is chosen slightly smaller than the recommended interval $\frac{N}{10} < d < \frac{N}{3}$ as the sensitivity of hoDMD results to d is very low, see [14].

In the end, eight DMD modes are chosen for the final representation of the given state-space trajectory based on the requirement $e_{\text{rrmse}}(\mathbb{X}^{\text{rec}}) < 0.2$ for the standard DMD, where relative RMS reconstruction error is computed as

$$e_{\text{rrmse}}(\mathbb{X}^{\text{rec}}) = \sqrt{\frac{\sum_{k=1}^N \|\mathbf{x}_k - \mathbf{x}_k^{\text{rec}}\|_2^2}{\sum_{k=1}^N \|\mathbf{x}_k\|_2^2}} \quad (7)$$

and $\mathbb{X}^{\text{rec}}$ is obtained by the application of formula (4). In order to be able to compare the performance of all the tested DMD variants, eight modes are chosen for rDMD, tIsDMD and hoDMD as well.

The DMD identified two most dominant DMD modes depicted on the right hand side of Figure 2. The left one with $\text{Im}(\omega_i) = 168.84$ Hz represents the dominant frequency of VF vibration where the right DMD mode without frequency content represents the dataset asymmetry. Both DMD modes have $\text{Re}(\omega_i) > 0$, which agrees with the DMD analysis of flutter, i.e. unstable phenomenon.

According to the magnitude of mode amplitude the most contributing mode has frequency of 168.84 Hz that lies between the second (156.03 Hz) and the third eigenfrequency (180.53 Hz) obtained by modal analysis (MA), see the results of [12]. Also its modal shape is highly similar to the third MA mode. We suppose that this DMD frequency is correct because according to [1, 2], the flutter occurrence is typically connected with the merge of two eigenfrequencies under external aerodynamic forcing. The combination of the second and the third structural eigenfrequencies in the phonation onset was reported e.g. by [16]. On the other hand the determination of the growth (or damping) rate is in general difficult and the value 8.65 provided by the DMD should be considered as a good estimate indicating a rapid flutter onset.

Although the growth rate of the asymmetric DMD mode is small (0.26) it is the second most contributing mode based on the used selection criterion and therefore it is important for accuracy of the dataset reconstruction. Its modal shape highly resembles the first mode obtained by modal analysis (MA).

Figure 2: **Left:** The approximation of continuous-time eigenvalue spectrum for three DMD variants. The rDMD spectrum is excluded as it is recomputed in every step of the rDMD algorithm. **Right:** Two most dominant DMD modes with their complex eigenfrequency obtained by the standard DMD. The left DMD mode has significant growth rate and frequency content of 168.84 Hz. The right DMD mode has no frequency content. The black frame illustrates the reference state with marked location of point A with coordinates [4.99; -0.505] mm.

Note. Any DMD mode with nonzero frequency content is complex and it appears in complex conjugated pairs. Figures 2 and 5 show only the real part of one DMD mode from the complex conjugated pair. Nevertheless, the DMD mode pair is still counted as two modes used for the final dataset reconstruction in the cases of DMD, tlsDMD and hoDMD. As the rDMD produces only real modes, each of them is counted as only one mode.

4.3 Comparison of DMD variants

The continuous-time spectra of the presented DMD methods are compared on the left hand side of Figure 2. It can be seen that eigenvalues of the tlsDMD and the hoDMD are shifted from complex half-plane with negative real part towards the imaginary axis in comparison with the standard DMD. This facilitates further computations because more relevant modes are available for final reconstruction due to the fact that term $\exp(\omega_i t)$ in formula (4) during long time interval for eigenvalues with too small as well as too big real part becomes meaningless. Especially, several eigenvalues with too large real part overflowed during the reconstruction step in the DMD and the tlsDMD which was corrected by excluding them as unphysical.

The results of all the DMD variants are summarized in Table 1. All the methods found the same dominant frequency present in the given dataset with similar growth rate. The sole exception is the hoDMD, where the growth rate is circa half. The modal shape of the mode with frequency ≈ 169 Hz for all DMD variants is nearly identical, see Figure 5. Furthermore, all methods also found the asymmetric mode to be significantly present in the given data. Nevertheless, its growth

rate substantially varies from positive values (in the DMD and rDMD cases) to negative ones (tlsDMD and hoDMD). The corresponding modal shapes are compared in Figure 6, where only the rDMD method provided the asymmetric mode more resembling the first rDMD mode than the first MA mode as others.

The computed magnitudes of modal amplitude are quite similar in cases of DMD and tlsDMD, while value z_1 in the rDMD case is circa twice as much as in the DMD case. This can be caused by retaining of one real rDMD mode instead of a complex conjugated pair as in other cases. The different values of z_i in the hoDMD case are caused by different normalization of the hoDMD modes.

interval	ω of the dominant mode	z_1	ω of 2. mode	z_2	e_{rmse}
DMD	$8.65 + 168.84 \cdot 2\pi \cdot i$	58.90	$0.258 + 0 \cdot i$	43.33	0.176
rDMD	$8.45 + 168.82 \cdot 2\pi \cdot i$	100.27	$0.085 + 0 \cdot i$	33.05	0.0059
tlsDMD	$9.57 + 168.76 \cdot 2\pi \cdot i$	61.35	$-0.027 + 0 \cdot i$	37.18	0.229
hoDMD	$4.39 + 168.94 \cdot 2\pi \cdot i$	0.889	$-0.00368 + 0 \cdot i$	0.854	0.0893

Table 1: Comparison of results of four DMD variants. The modes significance is ordered by the magnitude of modal amplitude denoted as z_i .

The quantitative comparison of the different DMD variants is accessed by the computation of the rmse error. Here the rDMD method outperformed all the others. The second place and approximately half rmse error compared to the DMD belongs to the hoDMD. The reconstruction accuracy is also documented by trajectory reconstruction in the point A , see Figure 3. Other DMD methods overestimate (DMD) or underestimate (tlsDMD, hoDMD) the final amplitude increase while rDMD tracks the original signal precisely. We can speculate about the reasons. The rDMD modes are by construction purely real and they do not appear in complex conjugated pairs as in other DMD methods. This probably provides possibility to retain more frequency content in the limited number of DMD modes. Moreover the possibility to violate the reconstruction formula (4) – as indicated in Figure 4, can be another important component allowing the more precise trajectory tracking; however, simultaneously losing the ability to use formula (4) for future state prediction.

Figure 3: Original signal (VF displacement in x -direction in point A) compared with four different DMD reconstructions. The second half of the input time interval is shown where the reconstruction correspondence gradually decreases.

5 Conclusion

The original DMD method and its three enhanced variants, namely the recursive DMD, the total least-square DMD and the higher order DMD are described. The rDMD is an interesting alternative combining advantages of POD (orthogonality) and DMD (mono-frequency modes). The tlsDMD and the hoDMD are DMD generalizations primarily developed to overcome the high

Figure 4: Time behaviour of the first four rDMD mode amplitudes. Second to fourth mode amplitudes do not follow the form of $b_i \exp(\omega_i t)$.

Figure 5: DMD modes with frequency content ≈ 169 Hz. From left: DMD, rDMD, tlsDMD and hoDMD.

Figure 6: Dominant asymmetric DMD modes. From left: DMD, rDMD, tlsDMD and hoDMD.

DMD sensitivity to noise. Moreover, the hoDMD enables the final DMD reconstruction to have an independent spatial and spectral complexity, see [10].

All four DMD variants are compared on the dataset of 2500 time steps of the phonation onset simulation. All methods identified similar two most dominant DMD modes where the main difference lies in different growth rates. In all the cases the main oscillatory DMD mode has the same frequency of ≈ 169 Hz and it is responsible for the flutter occurrence as its growth rate is clearly positive ($\text{Re}(\omega) \gg 0$). The overall reconstruction of the given dataset restricted to use of 8 DMD modes is the best for the rDMD, the hoDMD performs twice better than the DMD and the tlsDMD is worse than the DMD. We hypothesize that the rDMD outperforms the other methods due to violating the reconstruction formula given by Eq. (4). The computational demands of the rDMD and the hoDMD highly exceed the DMD due to the repetitive DMD applications in an iterative greedy loop and due to additional SVDs of multiply larger matrices, respectively. On the other hand they are not prohibitive. In fact, they increase the computational time approximately by an order of a magnitude (in the current study from a minute (DMD) to several minutes). The rDMD and the hoDMD decompositions are suitable for a further simplified modelling of phonation onset with prescribed parametrized VF motion as their reconstruction error is sufficiently low.

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