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A LAGRANGIAN APPROACH FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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Abstract

The motion of a viscous incompressible fluid flow in bounded domains with a smooth boundary can be described by the nonlinear Navier-Stokes system (N). This description corresponds to the so-called Eulerian approach. We develop a new approximation method for (N) in both the stationary and the nonstationary case by a suitable coupling of the Eulerian and the Lagrangian representation of the flow, where the latter is defined by the trajectories of the particles of the fluid. The method leads to a sequence of uniquely determined approximate solutions with a high degree of regularity, which contains a convergent subsequence with limit function v such that v is a weak solution on (N).

Keywords: Navier-Stokes Equations, Weak Solutions, Lagrangian Approach

1 Introduction

For the description of fluid flow there are in principle two approaches, the Eulerian approach and the Lagrangian approach. The first one describes the flow by its velocity

$$v = (v_1(t,x),\, v_2(t,x),\, v_3(t,x)) = v(t,x)$$

at time t in every point $x = (x_1, x_2, x_3)$ of the domain G containing the fluid. The second one uses the trajectory $x = (x_1(t),\, x_2(t),\, x_3(t)) = x(t) = X(t, 0, x_0)$ of a single particle of fluid, which at initial time $t = 0$ is located at some point $x_0 \in G$. The second approach is of great importance for the numerical analysis and computation of fluid flow also involving different media with interfaces [2, 3, 5, 8], while the first one has also often been used in connection with theoretical questions [4, 6, 7, 9].

It is the aim of the present note to develop a new approximation method for the nonlinear Navier-Stokes equations by coupling both the Lagrangian and the Eulerian approach. The method avoids fixpoint considerations and leads to a sequence of approximate systems, whose solution is unique and has a high degree of regularity, important at least for numerical purposes. Moreover, we can show that our method allows the construction of global weak solutions of the Navier-Stokes equations (compare [2, 4] for a local theory): The sequence of approximate solutions has at least one accumulation point satisfying the Navier-Stokes equations in a weak sense [6].

2 The Stationary Navier - Stokes Equations

We consider the stationary motion of a viscous incompressible fluid in a bounded domain $G \subset \mathbb{R}^3$ with a sufficiently smooth boundary S. Because for steady flow the streamlines and the trajectories of the fluid particles coincide, both approaches mentioned above are correlated by the autonomous system of characteristic ordinary differential equations

$$x'(t) \,=\, v(x(t)), \quad x(0) \,=\, x_0 \in G, \tag{1}$$

which is an initial value problem for

$$t \longrightarrow x(t) = X(t, 0, x_0) = X(t, x_0)$$

if the velocity field $x \longrightarrow v(x)$ is known in G.

To determine the velocity, in the present case we have to solve the steady-state nonlinear equations

$$\begin{aligned} -\nu\Delta v + v\cdot\nabla v + \nabla p &= F \quad \text{in} \quad G, \\ \operatorname{div} v = 0 \quad \text{in} \quad G, \qquad v &= 0 \quad \text{on} \quad S \end{aligned} \tag{2}$$

of Navier-Stokes. Here $x \longrightarrow p(x)$ is an unknown kinematic pressure function. The constant $\nu > 0$ (kinematic viscosity) and the external force density F are given data. The incompressibility of the fluid is expressed by $\operatorname{div} v = 0$, and on the boundary S we require the no-slip condition $v = 0$.

3 The Lagrangian Approach

Let us start recalling some facts, which concern existence and uniqueness for the solution of the initial value problem (1): If the function v belongs to the space $C_0^{\mathrm{lip}}(\overline{G})$ of vector fields being Lipschitz continuous in the closure $\overline{G} = G \cup S$ and vanishing on the boundary S, then for all $x_0 \in G$ the solution

$$t \longrightarrow x(t) = X(t, x_0)$$

is uniquely determined and exists for all $t \in \mathbb{R}$ (because $v = 0$ on the boundary S, the trajectories remain in G for all times). Due to the uniqueness, the set of mappings

$$\Re = \{X(t,\cdot) :\, G \to G |\, t \in \mathbb{R}\}$$

defines a commutative group of C^1- diffeomorphisms on G. In particular, for $t \in \mathbb{R}$ the inverse mapping $X(t,\cdot)^{-1}$ of $X(t,\cdot)$ is given by $X(-t,\cdot)$, i.e.

$$\begin{aligned} X(t,\cdot) \circ X(-t,\cdot) &= X(t,\, X(-t,\cdot)) \\ &= X(t-t,\cdot) = X(0,\cdot) = \mathrm{id}, \end{aligned}$$

or, equivalently,

$$X(t,\, X(-t,\, x)) = x$$

for all $x \in G$. Moreover we obtain $\det\nabla X(t,\, x) = 1$ if

$$v \in C_{0,\sigma}^{\mathrm{lip}}(\overline{G}) = \{u \in C_0^{\mathrm{lip}}(\overline{G}) |\, \operatorname{div} u = 0\},$$

in addition. This important measure preserving property implies

$$\langle f,\, g\rangle = \langle f \circ X(t,\cdot),\, g \circ X(t,\cdot)\rangle$$

for all functions $f, g \in L^2(G)$, where $\langle\cdot,\,\cdot\rangle$ denotes the scalar product in $L^2(G)$.

4 The Eulerian Approach

Next let us consider the Navier-Stokes boundary value problem (2). It is well known that, given $F \in L^2(G)$, there is at least one function v satisfying (2) in some weak sense [6]. To define such a weak solution we need the space $V(G)$, being the closure of $C_{0,\sigma}^{\infty}(G)$ (smooth divergence free vector functions with compact support in G) with respect to the Dirichlet-norm $\|\nabla u\| = \sqrt{\langle\nabla u,\, \nabla u\rangle}$, where we define

$$\langle\nabla u,\, \nabla v\rangle = \sum_{i,j=1}^{3} \langle D_j u_i,\, D_j v_i\rangle.$$

Let us recall the following

Definition 1 *Let $F \in L^2(G)$ be given. A function $v \in V(G)$ satisfying for all $\Phi \in C^{\infty}_{0,\sigma}(G)$ the identity*

$$\nu\langle\nabla v, \nabla\Phi\rangle - \langle v \cdot \nabla\Phi, v\rangle = \langle F, \Phi\rangle \tag{3}$$

is called a weak solution of the Navier-Stokes equations (2), and (3) is called the weak form of (2).

For a suitable approximation of the nonlinear term let us keep in mind its physical deduction. It is a convective term arising from the total or substantial derivative of the velocity vector v. Thus it seems to be reasonable to use a total difference quotient for its approximation.

To do so, let $v \in C^{lip}_{0,\sigma}(\overline{G})$ be given. Then for any $\varepsilon \in \mathbb{R}$ the mapping $X(\varepsilon, \cdot) : G \to G$ and its inverse $X(-\varepsilon, \cdot)$ are well defined. Consider for some $u \in C^1(G)$ ($C^m(G)$ is the space of continuous functions having continuous partial derivatives up to and including order $m \in \mathbb{N}$ in G) and $x \in G$ the one-sided Lagrangian difference quotients

$$\begin{aligned} L^{\varepsilon}_{+}\, u(x) &= \frac{1}{\varepsilon}\left[u(X(\varepsilon, \cdot)) - u(x)\right], \\ L^{\varepsilon}_{-}\, u(x) &= \frac{1}{\varepsilon}\left[u(x) - u(X(-\varepsilon, \cdot))\right], \end{aligned}$$

and the central Lagrangian difference quotient

$$L^{\varepsilon}\, u(x) = \frac{1}{2}\left(L^{\varepsilon}_{+}\, u(x) + L^{\varepsilon}_{-}\, u(x)\right). \tag{4}$$

Since for sufficiently regular functions

$$L^{\varepsilon}_{-}\, u(x) \longrightarrow v(x) \cdot \nabla u(x)$$

and

$$L^{\varepsilon}_{+}\, u(x) \longrightarrow v(x) \cdot \nabla u(x)$$

as $\varepsilon \to 0$, the above quotients can all be used for the approximation of the convective term $v \cdot \nabla v$. There is, however, an important advantage of the central quotient (4) with respect to the conservation of the energy:

Let $v \in C^{lip}_{0,\sigma}(G)$ and $u, w \in L^2(G)$. Let $X(\varepsilon, \cdot)$ and $X(-\varepsilon, \cdot)$ denote the mappings constructed from the solution of (1). Then, using the measure preserving property from above, we obtain only for the central quotient the orthogonality relation

$$\langle L^{\varepsilon}\, u, u\rangle = 0. \tag{5}$$

5 The Stationary Approximate System

To establish an approximation procedure we assume that some approximate velocity field v^n has been found. To construct v^{n+1} we proceed as follows:

1) Construct $X^n = X(\frac{1}{n}, \cdot)$ and its inverse $X^{-n} = X(-\frac{1}{n}, \cdot)$ from the initial value problem

$$x'(t) = v^n(x(t)), \qquad x(0) = x_0 \in G. \tag{6}$$

2) Construct v^{n+1} and p^{n+1} from the boundary value problem

$$\begin{aligned} -\nu\Delta v^{n+1} + \frac{n}{2}[v^{n+1} \circ X^n - v^{n+1} \circ X^{-n}] + & \\ + \nabla p^{n+1} = F \quad & \text{in} \quad G, \\ \operatorname{div}\, v^{n+1} = 0 \quad & \text{in} \quad G, \\ v^{n+1} = 0 \quad & \text{on} \quad S. \end{aligned} \tag{7}$$

Concerning the existence and uniqueness for the solution of (6) and (7) we need the usual Sobolev Hilbert spaces $H^m(G)$, $m \in \mathbb{N}$, which denote the closure of $C^m(G)$ with respect to the norm $\|\cdot\|_{H^m}$ (see [1]). A main result is now stated in the following

Theorem 2 a) *Assume $v^n \in H^3(G) \cap V(G)$ and $F \in H^1(G)$. Then for all $x_0 \in G$ the initial value problem (6) is uniquely solvable, and the mappings*

$$X^n : G \to G, \qquad X^{-n} : G \to G$$

are measure preserving C^1- diffeomorphisms in G.
Moreover, there is a uniquely determined solution

$$v^{n+1} \in H^3(G) \cap V(G),\, \nabla p^{n+1} \in H^1(G)$$

of the equations (7).
The velocity field v^{n+1} satisfies the energy equation $\nu\|\nabla v^{n+1}\|^2 = \langle F,\, v^{n+1}\rangle$.

b) *Assume $v^0 \in H^3(G) \cap V(G)$ and $F \in H^1(G)$. Let (v^n) denote the sequence of solutions constructed in view of Part a). Then (v^n) is bounded in $V(G)$ i.e. $\|\nabla v^n\|^2 \le C_{G,F,\nu}$ for all $n \in \mathbb{N}$, where the constant $C_{G,F,\nu}$ does not depend on n. Moreover, (v^n) has an accumulation point $v \in V(G)$ satisfying (3), i.e. v is a weak solution of the Navier-Stokes equations (2).*

6 The Nonstationary Navier - Stokes Equations

Let us consider now the motion of a nonstationary viscous incompressible fluid flow in a bounded domain $G \subset \mathbb{R}^3$ with a sufficiently smooth boundary S. Without loss of generality, in this section we assume conservative external forces and consider the following Navier-Stokes initial boundary value problem:

Construct a velocity field $v = v(t,x)$ und some pressure function $p = p(t,x)$ as a solution of the system

$$\begin{array}{rcll} v_t - \nu\Delta v + \nabla p + v\cdot\nabla v &=& 0 & \text{in } G, \quad t>0, \\ \nabla\cdot v &=& 0 & \\ v &=& 0 & \text{on } S, \quad t>0, \\ v &=& v_0 & \text{for } t=0. \end{array} \qquad (N)$$

Here v_0 is a suitable prescribed initial velocity distribution.

The existence of a classical solution global in time of this problem without any smallness restriction on the data has not been proved up to now. Hence also a globally stable approximation scheme does not exist for this system. In order to construct classically solvable equations, as in the steady-state case, an approximation of the nonlinear convective term $v\cdot\nabla v$, which is responsable for the non-global existence of the solution, by means of a Lagrangian difference quotients seems to be reasonable.

In the following we show that the nonstationary Navier-Stokes system (N) can also be approximated by means of Lagrangian differences. The resulting approximate system (N_ε) is uniquely solvable, its solution exists globally in time, has a high degree of regularity and satisfies the nonstationary energy equation.

7 The Initial Value Problem

Let J be a compact time interval, and let $\tilde{v} \in C(J, H^3(G) \cap V(G))$ be a given velocity field being strongly H^3-continuous. Consider the initial value problem

$$\begin{array}{rcl} \dot{x}(t) &=& \tilde{v}(t,x(t)) \\ x(s) &=& x_0 \end{array}, \quad (s,x_0) \in J \times \overline{G} \qquad \text{(A)}$$

concerning the trajectory $x(t) = X(t, s, x_0)$ of a fluid particle, which at time $t = s$ is located at x_0 in $\overline{G}$. Due to well-known results on ordinary differential equations, as in the autonomous case, the uniquely determined general solution $X(t, s, x_0)$ of (A) exists for all times, and the mapping

$$X(t, s, \cdot) : \overline{G} \to \overline{G}, \qquad t, s \in J$$

is a measure preserving diffeomorphism with inverse function

$$X^{-1} = X(s, t, \cdot).$$

As in the stationary case we now approximate the time dependent nonlinear convective term $v(t, x) \cdot \nabla v(t, x)$ by a central Lagrangian difference quotient as follows:

$$\begin{aligned} &v(t, x) \cdot \nabla v(t_0, x) \sim \\ &\sim \frac{1}{2\varepsilon} \Big(v(t_0, X(t + \varepsilon, t, x)) - v(t_0 \, , X(t, t + \varepsilon, x)) \Big). \end{aligned} \tag{8}$$

Here $\sim$ means that for a sufficiently regular function v the right hand side converges to the expression on the left hand side as $\varepsilon \to 0$.

The main advantage of the central quotient in (8), which we denote by

$$\frac{1}{2\varepsilon} \, (v \circ X - v \circ X^{-1})$$

for abbreviation, is again the validity of an analogon to the orthogonality relation of Hopf [6]:

Using $\langle \cdot, \cdot \rangle$ as $L^2(G)$-scalar product Hopf obtains the global (in time) existence of weak solutions to the Navier-Stokes system (N) due to the important orthogonality relation

$$(v \cdot \nabla v, v) = 0, \qquad v \in V(G).$$

Using the measure preserving property of the mapping X, we analogously obtain

$$\begin{aligned} &\frac{1}{2\varepsilon} \, (v \circ X - v \circ X^{-1}, v) \ = \\ &\qquad = \ \frac{1}{2\varepsilon} \Big((v \circ X, v) - (v, v \circ X) \Big) \ = \ 0, \end{aligned}$$

which implies the validity of the energy equation for all sufficiently regular solutions of the approximate system, if central Lagrangian differences instead of one-sided quotients are used.

8 Time Delay and Compatibility at Initial Time

To avoid fixed-point considerations for the solution of the regularized approximate system – the velocity vector v as well as the mappings X are unknown – by means of a time delay we replace $v \cdot \nabla v$ by $\frac{1}{2\varepsilon} \, (v \circ X - v \circ X^{-1})$ with trajectories X constructed at earlier time points, where the velocity v is known already.

To do so, on the given time interval $[0, T]$ we define a time grid by

$$t_k = k \cdot \varepsilon, \ k = 0, \ldots, N \in \mathbb{N},$$

where $\varepsilon := \frac{T}{N} > 0$. Setting

$$X_k := X(t_k, t_{k-1}, x),$$

for $t \in [t_k, t_{k+1})$ we can use e.g. the approximation

$$v(t, x) \cdot \nabla v(t, x) \ \sim \ \frac{1}{2\varepsilon} \, (v(t, X_k) - v(t, X_k^{-1})). \tag{9}$$

To initiate this procedure we extend the initial value v_0 continuously to a start function

$$v_s \in C([-\varepsilon, 0], H^3(G) \cap V(G)).$$

Then, indeed, on the subintervals $[t_k, t_{k+1})$ we can successively construct the mappings X_k from the given velocity field v and vice versa. Nevertheless, we do not obtain a global on $[0, T]$ existing solution of a problem regularized by (9). This is due to a certain compatibility condition, which always occurs in parabolic problems at the corner of the space time cylinder:
For the unique construction of the mapping X_k, if integer order Sobolev spaces are used, we need a velocity field

$$v \in C([t_{k-1}, t_k], H^3(G) \cap V(G)),$$

i.e.

$$v_t \in C([t_{k-1}, t_k], V(G)).$$

Using

$$P : L^2(G) \to H(G) := \overline{C_{0,\sigma}^\infty(G)}^{\|\cdot\|}$$

as orthogonal projection we obtain in particular the condition

$$\begin{aligned} v_t(t_k) = {}& \mu P\Delta v(t_k) - \\ & -\frac{1}{2\varepsilon} P\Big((v(t_k, X_k) - v(t_k, X_k^{-1})\Big) \in V(G). \end{aligned} \tag{10}$$

Due to $v_0 \in H^3(G) \cap V(G)$ we find that the right hand side of (10) is contained in $H^1(G) \cap H(G)$, only. Hence the condition $v_t(t_k) \in V(G)$ implies in case of an approximation of the type (9) that we have to impose the condition

$$\begin{aligned} & \mu P\Delta v(t_k) - \\ & -\frac{1}{2\varepsilon} P\Big((v(t_k, X_k) - v(t_k, X_k^{-1})\Big) = 0 \quad \text{on} \quad S. \end{aligned} \tag{11}$$

9 The Approximate System (N_ε)

Instead of a system regularized by (9) we consider

$$\begin{array}{rcll} v_t - \mu\Delta v + \nabla p + Z_\varepsilon v & = & 0 & \multirow{2}{*}{in G for $t > 0$,} \\ \nabla \cdot v & = & 0 & \\ v & = & 0 & \text{on } S, \\ v_t & = & f & \text{in } G \text{ for } t = 0, \end{array} \tag{N_ε}$$

where $f \in V(G)$, and where for $t \in [t_k, t_{k+1}]$

$$\begin{aligned} Z_\varepsilon v(t, x) := \frac{1}{2\varepsilon}\Big(& (t - t_k)(v(t, X_k) - v(t, X_k^{-1})) + \\ & + (t_{k+1} - t)(v(t, X_{k-1}) - v(t, X_{k-1}^{-1}))\Big) \end{aligned}$$

is continuously defined on $[0, T]$.

In this case all compatibility conditions are satisfied: The condition for $t = 0$ can be fulfilled following a hint of V. A. Solonnikov by prescribing $v_t(0) = f \in V(G)$ instead of $v(0) = v_0$:

For a given function

$$v_s \in C([-2\varepsilon, -\varepsilon], H^3(G) \cap V(G))$$

we solve the problem (A) and obtain the mapping X_{-1}. Then we consider the stationary problem

$$\nu P\Delta v_0 - \frac{1}{2\varepsilon} P(v_0 \circ X_{-1} - v_0 \circ X_{-1}^{-1}) = f,$$

and obtain by well-known existence and regularity results a uniquely determined solution

$$v_0 \in H^3(G) \cap V(G),$$

which, since functions in $V(G)$ vanish on the boundary S, satisfies the required compatibility condition (11). By linear interpolation between $v_s(-\varepsilon)$ and v_0 we then obtain a start function

$$v_s \in C([-2\varepsilon, 0], H^3(G) \cap V(G)).$$

Since the compatibility condition in all the following grid points t_k are automatically satisfied due to the continuity of the function

$$t \to Z_\varepsilon v(t),$$

we finally obtain, by successively constructing the mappings fom the velocity field v and vice versa, the following result:

Theorem 3 *Let $[0,T]$ be given and let $f \in V(G)$ Then for every $\varepsilon > 0$ exists a uniquely determined function*

$$v \in C([0,T], H^3(G) \cap V(G))$$

und a uniquely determined pressure gradient

$$\nabla p \in C([0,T], H^1(G))$$

as the solution of the system (N_ε). For v holds on $[0,T]$ the energy equation

$$\|v(t)\|^2 + 2\nu \int_0^t \|v(s)\|^2 \, ds \; = \; \|v_0\|^2,$$

and H^3–Norm estimates can be constructed uniformly on $[0,T]$ depending on the data, T and ε.

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